ELECTRICITY

AND

MAGNETISM.

A MATHEMATICAL TREATISE FOR ADVANCED UNDERGRADUATE STUDENTS.

BY

FRANCIS E. NIPHER, A. M.,

Professor of Physics and in charge of Electrical Engineering in Washington University, St. Louis, Mo.

ST. LOUIS, MO., U. S. A.:

JOHN L. BOLAND BOOK AND STATIONERY CO.

1895.

NIXON-JONES PTG. CO., 215 PINE ST., ST. LOUIS.

PREFACE.

This book is intended to be a mathematical exposition of the fundamental principles of Electricity and Magnetism. It is designed for the use of students who have but recently begun to use the processes of the calculus, and it has been an incidental aim of the author to assist the pupil in acquiring possession of the machinery of mathematics. There has been no attempt to avoid any legitimate analytical method, because it is not popularly known, but on the other hand there has been an attempt to avoid wasting the time of the reader over puzzles and obscurities which are made difficult, and called easy.

All of the resulting equations are illustrated by numerical examples which are fully solved, so that those who do not care to follow the mathematics, may easily and safely use the results in the work of designing. Whenever possible, the numerical applications have been selected with reference to their use in the design of apparatus and machinery for specific work.

The arrangement of subjects is perhaps somewhat unusual. The aim has been to secure clearness, and preserve the logical unity of the treatment which has been attempted. This is deemed more important than that the subject-matter should be presented in an elaborate array of equally-spaced compartments.

The author has had the advantage of the careful criticism of Professor E. A. Engler of Washington

University, who has read the manuscript, and has given many valuable suggestions. It is the intention to make such changes and improvements in future editions as may seem desirable. Any criticisms intended to assist in securing this result will be thankfully received.

FRANCIS E. NIPHER.

WASHINGTON UNIVERSITY,
ST. LOUIS, June 14, 1895.

TABLE OF CONTENTS.

INTRODUCTION.

CHAPTER I.

ATTRACTION.

CHAPTER II.

POTENTIAL.

CHAPTER III.

ELECTROSTATICS.

CHAPTER IV.

CAPACITY.

CHAPTER V.

ENERGY.

CHAPTER VI.

MAGNETISM.

CHAPTER VII.

ELECTRIC CURRENTS.

CHAPTER VIII.

VARIABLE CURRENTS WITH CONSTANT POTENTIAL SOURCE.

CHAPTER IX.

ALTERNATING CURRENTS

CHAPTER X.

IRON CORES IN ELECTRO-MAGNETIC SYSTEMS.

CHAPTER XI.

UNITS.

CHAPTER XII.

PROBLEMS.

INTRODUCTION.

1. The fundamental units to which all physical measurements can be referred, are those of length, mass and time. The units adopted in this treatise are the centimetre, the gramme and the second.

2. The unit velocity is, therefore, one centimetre per second.

3. The unit acceleration is a change in velocity, of one centimetre per second. Acceleration is measured in centimetres per second, per second.

4. The unit force is the force which can change the acceleration of a gramme by unity, or can produce unit acceleration on a gramme. This unit is called the dyne. The earth's attraction on a gramme produces an acceleration usually denoted by g. The number of units of force in the weight of a gramme is, therefore, g dynes. At the earth's surface g is about 981 dynes.

5. A field of force is any region in which a testing mass tends to move. If such motion be unconstrained, the motion will be in a definite direction, called the direction of the force. If the motion be constrained by an interposed frictionless plane, upon which the testing mass may slide, it will slide in a definite direction on this plane. That direction is the one which deviates least from the direction in unconstrained motion. In this direction the acceleration is greater than it would be along any other path on the plane. It is the direction of greatest slope. The intensity of the field at any point is measured by the force in dynes acting on a unit

mass at the point and tending to produce motion in the direction of maximum acceleration. If the field be uniform the lines of direction through the points of the field are parallel, as will clearly appear in later discussions.

It is customary to represent the intensity of a field by drawing lines in the direction of the force, the number of lines to the square centimetre being equal to the number of dynes acting on a unit of mass at the point. The component of the field in any direction is expressed in the same manner. On a plane at right angles to the direction of unconstrained motion, the number of lines to the square centimetre would be zero. To represent the earth's gravitation field, the lines of force would be vertical, the number of lines to the square centimetre being g.

Gravitation lines of force terminate in matter. Electrical lines of force terminate in electrical charges. Magnetic lines of force terminate in magnetized masses, or link with electric currents.

6. The unit of energy is the dyne-centimetre or erg. This is the energy required to exert a force of one dyne over a distance of one centimetre.

7. The unit of power is the erg per second. It is the power required to apply work at the rate of one erg per second. Taking the weight of a kilogramme at the earth's surface as the unit of force and assuming the weight of a gramme, g, to have an average value of 981 dynes, this unit of force would be 981,000 or 9.81×10^5 dynes. The horse-power is —

76 kilogramme-metres per second,
or 7.6×10^6 gramme-centimetres per second,
$7.6 \times 10^6 \times 981 = 7.4556 \times 10^9$ ergs per second.

8. *Magnitude of Physical Units.*

Let L, M, T represent respectively the magnitude of the centimetre, the gramme and the second, and let L', M' and sent the magnitude of the foot, the pound and Then if a wire is found to be l centi-

metres or l' feet in length, its length may be represented either as $l\,L$ or $l'\,L'$ and hence

$$l\,L = l'\,L',$$
$$\text{or} \quad l = \frac{L'}{L}\,l'.$$

The ratio $\frac{L'}{L}$ is the length of a foot in centimetres. Direct comparison shows this value to be 30.4797. Hence any number l' of feet would be equal to $30.4797l'$ centimetres.

Similarly, if m and m' represent the number of units of mass in any piece of matter in the two systems of units

$$m = \frac{M'}{M}\,m'$$

where the conversion factor $\frac{M'}{M}$ is the number of grammes in a pound or 453.59.

Area is the product of two linear dimensions. The product of length and breadth of a rectangular area in square centimetres is

$$l\,L \times b\,L = l\,b\,L^2,$$

where L^2 represents the magnitude of the unit area in turns of the centimetre. For converting an area s' square feet into the equivalent s centimetres,

$$s = \left(\frac{L'}{L}\right)^2 s'.$$

Similarly for volumes

$$v = \left(\frac{L'}{L}\right)^3 v'.$$

Density is defined as mass per unit volume. The unit of density is such a density that a unit mass is contained in a unit of volume. The magnitude of the unit density must, therefore, depend on the magnitudes of the units

of mass and of volume. Water at $4°C$ has a density of one gramme to the cubic centimetre. Expressed in this set of units, the density of water is unity. If the kilogramme be taken as the unit of mass, then the unit density would be one kilogramme per cubic centimetre. The density of water in this system is 0.001. In gramme-decimetre units, the unit density would be one gramme per cubic decimetre. In these units the density of water would be 1,000. In kilogramme-decimetre units the density of water would again be unity. The density of cast iron is 7.207 in gramme-centimetre units and 449.86 in pound-foot units.

Let D be the magnitude of the unit density in centimetre-gramme units, and D' the magnitude in foot pound units. Suppose we measure the dimensions of a rectangular block of iron, and find them to be l, b, h centimetres. Then we may write the equation for density of iron

$$d.\ D = \frac{m\ M}{l\ b\ h.\ L^3}.$$

If $l = 10$, $b = 5$, $h = 2$ centimetres and we find by the balance that $m = 720.7$ grammes, then

$$7.207\ D = \frac{720.7\ M}{2 \times 5 \times 10\ L^3}.$$

It follows that

$$D = \frac{M}{L^3}.$$

If we measure another block of the same iron and find its dimensions to be $l' = 5$, $b' = \frac{1}{2}$, $h' = 4$ feet and we find by the balance that $m' = 4498.6$ lbs., then

$$449.86\ D' = \frac{4498.6\ M'}{5 \times \frac{1}{2} \times 4\ L'^3}.$$

Hence

$$D' = \frac{M'}{L'^3}.$$

It is evident that D and D' are not equal, but that for iron

$$7.207\ D = 449.86\ D',$$

or

$$\frac{720.7\ M}{2 \times 5 \times 10\ L^3} = \frac{4498.6\ M'}{5 \times \frac{1}{2} \times 4\ L'^3},$$

$$\therefore \qquad 7.207\ \frac{M}{L^3} = 449.86\ \frac{M'}{L'^3};$$

or

$$7.207 = \frac{M'}{M}\left(\frac{L}{L'}\right)^3 449.86.$$

In general

$$d = \frac{M'}{M}\left(\frac{L}{L'}\right)^3 d'$$

$$= 453.57\left(\frac{1}{30.4797}\right)^3 d'$$

Therefore if M represents the magnitude of the unit of mass in any system, and L that of unit length, the magnitude of the unit of density may be expressed in terms of those units as

$$D = \frac{M}{L^3}.$$

If the unit of mass be M' and of length L' the magnitude of the unit of density will be

$$D' = \frac{M'}{L'^3}.$$

The magnitude of the unit of density does not depend upon the unit of time. A change from seconds to minutes does not affect the numerical values in which we represent densities. It does clearly appear that the

magnitude of the unit of density is directly proportional to that of the unit of mass, and inversely as that of the cube of the unit length.

Similar explanations apply in succeeding cases. It is important to observe the way in which the unit in which any physical quantity is measured depends on the three fundamental units, in order that the effect of a change of units may be determined.

$$\textit{Velocity } v = \frac{l}{t}.$$

The magnitude of the unit of velocity is therefore $\frac{L}{T}$, and

$$\therefore v = \frac{L'}{L}\,\frac{T}{T'}\,v'.$$

$$\text{where } \frac{T}{T'} = \frac{\text{second}}{\text{minute}} = \frac{1}{60}.$$

If the units of time were the same, the conversion factor for determining the number of centimetres per second in v' feet per second would be $\frac{L'}{L}$.

$$\text{Momentum} = m\,v. = m\,\frac{l}{t}.$$

The magnitude of the unit is

$$M\,\frac{L}{T} = M\,L\,T^{-1}$$

$$\therefore m\,v = \frac{M'}{M}\,\frac{L'}{L}\,\frac{T}{T'}\,(m\,v)'.$$

$$\textit{Acceleration } a = \frac{v}{t} = \frac{l}{t^2}.$$

The magnitude of the unit is $L\,T^{-2}$.

$$\therefore a = \frac{L'}{L}\left(\frac{T}{T'}\right)^2 a'.$$

$$\textit{Force} = m\,a = \frac{m\,v}{t} = \frac{m\,l}{t^2}.$$

The magnitude of the force unit is

$$M\,L\,T^{-2}$$

$$\therefore\ f = \frac{M'}{M}\,\frac{L'}{L}\left(\frac{T}{T'}\right)^2 f'$$

The unit of force is called the dyne.

$$Work = f\,l = \frac{m\,l^2}{t^2}\left(\text{or } \frac{m\,v^2}{2}\right)$$

$$\therefore\ w = \frac{M'}{M}\left(\frac{L'}{L}\right)^2\left(\frac{T}{T'}\right)^2 w'$$

The unit of work is the dyne-centimetre or erg.*

Power or *Time Rate of Work.*

$$P = \frac{f\,l}{t} = \frac{m\,l^2}{t^3}$$

The magnitude of the unit is

$$M\,L^2 T^{-3}$$

$$\therefore\ P = \frac{M'}{M}\left(\frac{L'}{L}\right)^2\left(\frac{T}{T'}\right)^3 P'.$$

If work is expressed in foot-pounds and kilogramme-metres, the unit of force being taken as the weight of a pound and of a kilogramme at the earth's surface, or

* The expression for kinetic energy is $\frac{m\,v^2}{2}$. The numeric $\frac{1}{2}$ in this expression is independent of magnitudes of space, mass and time units. The magnitude of the unit in which a numeric, or, a ratio of like quantities, as π, *sin*, *tan*, etc., are measured, must be considered unity. Their numerical values are the same in foot-grain-second units as in *C. G. S.* units. The conversion factor for changing from any system of units to any other, is, therefore, unity for quantities of this character. If we choose to represent the magnitude of the unit in which a numerical quantity is measured, in terms of *C. G. S.* units, the magnitude of such unit will be $M^0\,L^0\,T^0 = 1$, since any quantity to zero power is unity. This is equivalent to saying that this unit is independent of M, L, and T. The magnitude of the unit in which an angle $= \frac{arc}{radius}$ is measured is unity, in precisely the same sense that the magnitude of the unit area is L^2.

more accurately, at some point on the earth's surface, then

$$w = \frac{M'}{M}\frac{L'}{L}w',$$

$$\text{where } \frac{M'}{M} = \frac{\text{pound}}{\text{kilogramme}} = 0.4536;$$

$$\frac{L'}{L} = \frac{\text{foot}}{\text{metre}} = 0.3048.$$

The conversion factor being thus 0.1383.

Similarly, to convert foot-pounds per minute into kilogrammetres per second,

$$P = \left(\frac{M'}{M}\frac{L}{L}\frac{T}{T'}\right)P',$$

where the conversion factor becomes 0.000230.

The horse-power is 33,000 times as great as the unit in which P' is measured, and the *force de cheval* is 76 times as great as the unit in which P is measured. Hence if P is to be in *force de cheval* and P' in horse-power, the equation will be

$$P = 0.000230\frac{33000}{76}P'$$

$$\text{or } P = 1.0139\,P'.$$

$\therefore$ one horse-power = 1.014 *force de cheval.*

9. *Electrical Quantity.*

Electrical quantity is measured by two methods. In both methods the determination is based upon the measurement of a force exerted by the electricity, as we determine masses of ordinary matter by means of its weight. A mass of matter upon which the earth's pull is ten times as great as upon the standard unit of mass, has a mass ten times as great as the unit mass.

10. *The Electrostatic Unit of Quantity* is the quantity of electricity which, acting on an equal quantity of elec-

tricity at a distance of one centimetre, will repel it with a force of one dyne.

As we shall see, a small spherical mass of 3928 grammes of matter, will attract an equal mass at any distance r, with the same force that an unit charge of electricity on each of them will cause them to repel each other. (Sections 21 and 220.)

11. *Magnitude of the Electrostatic Unit of Quantity.*

The unit of quantity being indicated by Q, any electrical quantity may be represented by $q. Q$. The force with which two quantities q and q' act on each other at any distance r is known by experiment to be

$$f = \frac{q\,q'}{r^2}.$$

Expressing in symbols the units that are always understood, this equation becomes —

$$f\frac{M\,L}{T^2} = \frac{q\,Q \times q'\,Q}{r^2\,L^2}.$$

Dividing by the common factor we have,

$$\frac{M\,L}{T^2} = \frac{Q^2}{L^2}$$

$$\therefore\ Q = M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-1}$$

which gives the magnitude of the electrostatic unit of quantity in terms of fundamental units.

12. The unit current of electricity is defined to be one in which unit quantity flows per second. Therefore the electrostatic unit of current is

$$I = \frac{Q}{T} = M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-2}$$

13. The unit magnet pole is defined precisely like unit quantity of electricity, and its magnitude is the same as that of Q.

14. The other method of measuring electrical quantity is based on the magnetic effects of electricity in motion. The unit current is thus primarily defined, and the unit quantity is deduced therefrom.

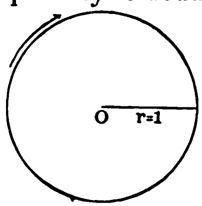


Fig. 1.

It is known by experiment that if a current of electricity flow around a circular circuit (Fig. 1) having a radius of one centimetre, that a magnet pole at the centre is urged along the axis of the circle. If the current flow in a clockwise direction as in Fig. 1, a north pole at O would be urged away from the reader, at right angles to the paper. Reversing the current, reverses the direction of the force. The intensity of the force acting on unit pole is directly proportional to the current flowing. This being established by experiment, if we adjust the current so that unit pole is acted upon by a force of 2π dynes, then each unit of length of the current will act with a force of one dyne. The current will then be unit current.

The equation which experiment shows represents the force f with which any circular current i of radius r acts upon a pole m at its centre, is

$$f = K \frac{2\pi r i m}{r^2}.$$

The force is directly as the length of the equidistant circuit $2\pi r$, to the strength of current and pole, and inversely as the square of the radius. If as stated we adjust i so that when $m = 1$ and $r = 1$, $f = 2\pi$ dynes, then,

$$i = \frac{1}{K}.$$

If we now take this as the unit current, then we make $K = 1$, and measured in these units

$$f = \frac{2\,\pi\, i\, m}{r},$$

or

$$i = \frac{r\,f}{2\,\pi\, m}$$

It is apparent from this equation that the magnitude of the electro-magnetic unit of current must be directly proprotional to the magnitude of the force unit and the length unit, and inversely as that of unit magnet pole.

Denoting unit current in electro-magnetic measure by $\overline{I}$ we have

$$\overline{I} = \frac{M\,L\,T^{-2}\,L}{M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-1}} = M^{\frac{1}{2}}\,L^{\frac{1}{2}}\,T^{-1}.$$

It will be observed that the square of this unit current is a force unit. The electro-magnetic unit of quantity is, therefore,

$$\overline{Q} = \overline{I}\,\overline{T} = M^{\frac{1}{2}}\;L^{\frac{1}{2}}.$$

This unit is, therefore, independent of the time unit.

15. Density of electrical charge or of magnetism is quantity per unit area. The magnitude of its unit in electrostatic measure is,

$$D = \frac{Q}{L^2} = M^{\frac{1}{2}}\,L^{-\frac{1}{2}}\;T^{-1}.$$

16. Unit Field. In a field of strength H the force acting on a pole m or a quantity of electricity q is known by experiment to be

$$f = m\,H \text{ or } q\,H.$$

Therefore the magnitude of unit electric or magnetic field is

$$\frac{F}{Q}=\frac{M\,L\,T^{-2}}{M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-1}}=M^{\frac{1}{2}}L^{-\frac{1}{2}}\;T^{-1}.$$

It will be observed that strength of field and density of charge depend on the units of mass, length and time, in precisely the same way.

17. The magnitudes of other units will be shown as the equations in which they appear are developed. It will be seen that every equation in which a physical quantity enters, must define the unit in which that quantity is measured in terms of the units of other quantities found in the equation. It should also be remembered, that in every physical equation, terms which are added together must have the same unit when referred to the fundamental units, and that when all the symbols representing concrete quantities are transferred to one side of an equation, leaving numerics only in the other member as in the equation

$$f\,(x,\,y,\,z)=a,$$

the unit of the left hand member must be unity. An equation which does not satisfy this test is erroneous.

18. *Approximation formulæ.*

If x and y represent any variable quantities and $\delta\,x$ and $\delta\,y$ represent small terms, then in any expression

$$z=(x+\delta\,x)\quad(y+\delta\,y),$$

we may write

$$z=x\,y\left(1+\frac{\delta\,x}{x}\right)\quad\left(1+\frac{\delta y}{y}\right)$$

$$=x\,y\left(1+\frac{\delta\,x}{x}+\frac{\delta\,y}{y}\right)$$

$$= x\,y + x\,y\ \left(\frac{\delta\,x}{x} + \frac{\delta\,y}{y}\right),$$

Similarly, if

$$z = \frac{y + \delta\,y}{x + \delta\,x}$$

$$\text{then } z = \frac{y}{x}\ \frac{1 + \dfrac{\delta\,y}{y}}{1 + \dfrac{\delta\,x}{x}} = \frac{y}{x}\left(1 + \frac{\delta\,y}{y} - \frac{\delta\,x}{x}\right)$$

$$= \frac{y}{x} + \frac{y}{x}\ \left(\frac{\delta\,y}{y} - \frac{\delta\,x}{x}\right)$$

If $z = (y + \delta\,y)^n$

$$\therefore\ z = y^n\ \left(1 + \frac{\delta\,y}{y}\right)^n$$

$$= y^n\ \left(1 + n\ \frac{\delta\,y}{y}\right),$$

where n may be any integer or fraction.

19. *Measurement of Solid Angles.*

Let C be the centre of a sphere of radius R. Draw any diameter $O\ C$, and any radius R, making an angle a with $O\ C$. Revolve R about $O\ C$ keeping a constant. The end of R will describe a circle having a radius y. Repeat the operation with a increased by $d\ a$. We shall then have a zone of width $d\ s$ between the two circles traced by the end of the radius R.

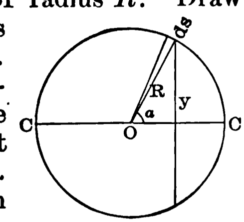


Fig. 2.

The area of this zone is

$$d\ A' = 2\ \pi\ yds.$$

where
$$y = R\ sin\ a$$
$$d\ s = R\ d\ a$$
$$\therefore\ d\ A' = 2\ \pi\ R^2\ sin\ a\ d\ a$$

The area of the spherical sector included within any cone of semi-angle a will be

$$A' = 2\,\pi\,R^2 \int_0^a \sin a\, d\, a$$

$$= 2\,\pi\,R^2 (1 - \cos a).$$

If $a = 180$, $\cos a = -1$ and the area of the sphere becomes $4\,\pi\,R^2$

Calling the value $\dfrac{A'}{R^2} = \omega$

$$\omega = 2\,\pi\,(1 - \cos a).$$

This is the area of a spherical sector on a unit sphere, included within a circular cone of semi-angle a. It will thus be seen that

$\dfrac{\textit{area of spherical sector}}{(\textit{radius})^2}$, is the area of the similar spherical sector on a sphere of unit radius, and measures the solid angle subtended by the spherical sector precisely as $\dfrac{\textit{arc}}{\textit{radius}}$ measures a plane angle.

CHAPTER I.

ATTRACTION.

20. *Law of Attraction.*

It has been established by experiment that the attraction between two small spherical masses, separated by a distance r, is

$$A = K \frac{m\,m'}{r^2} \quad \text{..................... (1).}$$

If the two bodies approach each other under this attraction the accelerations of the bodies being a and a' respectively, then the force A is also

$$A = K'\,m\,a = K'\,m'\,a' \quad \text{..............(2).}$$

In (1) if $m = m' =$ one gramme, and $r =$ one centimetre, then

$$A = K.$$

The constant K may be defined as the force with which unit mass will attract an equal mass distant one centimetre.

In (2) if $m = 1$ and $a = 1$

$$A = K'.$$

Therefore, K' may be defined as the force required to give unit acceleration to unit mass. The latter force has been universally adopted as the unit force, and in the C. G. S. system this unit is called the dyne. In this unit (2) becomes

$$A = m\,a = m'\,a' \quad \text{................(3).}$$

Equation (3) takes no account of the origin of the force acting upon a body, but measures the force in terms of its effects in producing, or changing motion. It is, however, evident that (3) will apply to the case of attraction between bodies represented by (1). If, for example, m represent the mass of the earth, m' the mass of a body falling freely in the earth's gravitation field, a' being then $g = 981$, then a would be the acceleration of the earth in falling towards the body m'; then

$$m\,a = m'\,g \qquad \text{(4)}.$$

Here $m'\,g$ is ordinarily called the weight of the body m'. In a precisely similar sense, $m\,a$ is the weight of the earth. Equation (4) asserts that the weight of the earth is equal to the weight of any gravitating body. If $m' = 3$ grammes, where $g = 981$, then the weight of m' is $3 \times 981 = 2943$ dynes, and the weight of the earth is also 2943 dynes.

A study of the earth's figure, and a determination of its mean density, has established the value of the earth's mass as $m = 6.14 \times 10^{27}$ grammes. Therefore, the acceleration of the earth in falling freely towards three grammes is

$$a = \frac{3 \times 981}{6.14 \times 10^{27}} = 4.79 \times 10^{-25}.$$

It will, therefore, be understood that the expression "weight of the earth" has no physical meaning. This weight may be anything. It is an indeterminate quantity.

21. *Astronomical Unit of Mass.*

Expressing A in dynes, the numerical value of K in (1) may be determined, by finding any simultaneous values of the other quantities.

If m' grammes be acted on by the earth, where the acceleration of a falling body is 981, then from (3) and (1)

$$m' \times 981 = K \frac{m'\,6.14 \times 10^{27}}{(6.37 \times 10^{8})^{2}},$$

where 6.37×10^8 is the mean radius of the earth in centimetres.

Hence,

$$K = \frac{981\ (6.37 \times 10^8)^2}{6.14 \times 10^{27}}$$

$$\text{or } K = \frac{1}{1.543 \times 10^7} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

It is for the present assumed that the attraction of the earth is the same as if its mass were concentrated in a heavy particle at its centre. The last equation, (5) represents the attraction in dynes of a gramme of matter condensed to a particle, on an equal mass, distant one centimetre. This expression in (1) will enable us to find the attraction between any two masses of matter, if the distance r is given.

We may also determine what mass must be condensed to a particle in order that its attraction on an equal mass distant one centimetre, may be one dyne. We have by (1)

$$1 = \frac{m \times m}{1.543 \times 107}$$

$$\text{or } m = \sqrt{1.543 \times 107} = 3928.$$

A mass of 3928 grammes is called the astronomical unit of mass, because if masses are expressed in this unit instead of in grammes, the astronomical equation of attraction becomes

$$A = \frac{m\,m'}{r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

where A and r are in dynes and centimetres respectively. (See section 10.)

22. *Attraction of a thin Circular Plate on a Particle in its Axis.*

Suppose a mass of 3928 grammes to be placed at O, in the axis of a thin plate Fig. 3 and distant h centimetres

from its centre. Then the mass of an annulus having radius r, a radial width dr, and a thickness dh is

$$d^2m = \frac{2\pi r\, dr\, dh\, \rho}{3928}$$

where ρ is the density in grammes to the cubic centimetre, and $\frac{\rho}{3928}$ is the density in astronomical units. Each particle of this ring is distant l from the attracted unit of mass at O.

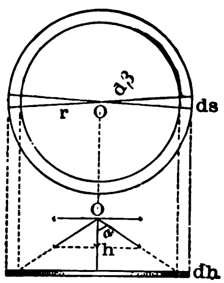


Fig. 3.

In order to find the attraction of this ring on the unit mass at O, consider the ring as made up of elements of the second order. In the plan view of Fig. 3 two diameters making with each other an angle $d\beta$ cut from the ring opposing elements, each having a volume $ds\, dr\, dh$ and a mass $\frac{\rho}{3928} \times ds\, dr\, dh$. The attraction of each of these elements on O, will be along the slant height l and will be $\frac{\rho}{3928}\,\frac{ds\, dr\, dh}{l^2}$. But these attractions oppose each other in part. For opposite elements, the components parallel to the plate will balance each other, and the same will be true for all the opposite elements composing the ring. The integral of this component will be zero. The component along the axis of the plate will be

$$\frac{\rho}{3928}\,\frac{ds\, dr\, dh}{l^2}\cos a$$

But $ds = r\, d\beta$. Substituting this value and integrating in β around the ring we have for the attraction of the elementary ring

$$d^2 A = \frac{\rho r\, dr\, dh}{3928\, l^2}\cos a \int_0^{2\pi} d\beta$$

$$= \frac{2\pi\, \rho r\, dr\, dh}{3928\, l^2}\cos a$$

Ex. A mountain having a height $h = 3$ miles $= 4.83 \times 10^5$ cm., and having a density 3.0 in grammes to the cubic centimetre, and a semi-angle α of 60°, the attraction on 3928 grammes at its vertex would be

$$\frac{\pi \rho h}{3928} = \frac{\pi \times 3 \times 4.83 \times 10^5}{3928} = 1159 \text{ dynes.}$$

The attraction of the earth on this mass is $981 \times 3928 = 3853368$ dynes.

24. *Special Cases.* Calling σ the quantity per unit area of the plate, $= \frac{\rho\, dh}{3928}$, and ω the solid angle subtended by the plate at the point O, and (7) becomes (section 19)

$$A = \sigma\,\omega \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

At the surface of the plate $\alpha = 90°$ and $\omega = 2\pi$. The attraction of the plate then becomes

$$A = \frac{2\pi\,\rho\, dh}{3928} = 2\,\pi\,\sigma \ldots\ldots\ldots\ldots (10).$$

If the plate be of infinite extent (10) would represent its attraction on a particle at any finite distance h from the plate. The plate may then have any finite thickness, and $\sigma = \frac{\rho\, h}{3928}$.

Equation (8) for the attraction of a solid cone becomes identical with the equation for attraction of an infinite plate of thickness equal to the height of the cone; if $\alpha = 90°$ in (8). The resulting attraction is

$$A = \frac{2\pi\,\rho h}{3928} \ldots\ldots\ldots\ldots\ldots (11).$$

25. The attraction of a circular cylinder on a unit *particle at the* centre of one end, is found by integrating

(7) in h, with r constant, $\cos a$ being replaced by $\dfrac{h}{\sqrt{h^2 + r^2}}$. Then

$$A = \frac{2\pi\rho}{3928}\left[\int_o^h dh - \int_o^h \frac{h\,dh}{\sqrt{r^2 + h^2}}\right]$$

$$= \frac{2\pi\rho}{3928}\left(h + r - \sqrt{r^2 + h^2}\right) \quad \ldots\ldots\ldots\ldots\ldots(12).$$

In order that the attraction of the cylinder may be one-half of the attraction of an infinite plate of the same thickness h, we have by (12) and (11)

$$2\left(h + r - \sqrt{r^2 + h^2}\right) = h$$

$$\text{or } r = \frac{3}{4}h$$

In general if the attraction of the cylinder is to be $\dfrac{1}{n}$ of the attraction of an infinite plate having the same thickness h, then

$$r = \frac{h}{2n}\,\frac{2n - 1}{n - 1}.$$

In case of a thin plate, the condition that a circular plate shall have $\frac{1}{2}$ the attraction of a plate of infinite extent is by (7) and (10)

$$\pi\sigma = 2\pi\sigma(1 - \cos a)$$

$$\text{or } \cos a = \frac{1}{2} \therefore a = 60°.$$

26. In Fig 4, O is the centre of the circular end of the cylinder, whose radius $= \dfrac{3}{4}h$ and whose attraction at O is half that of the infinite plate of thickness h. It is easily shown that the attraction of a cone whose *height is h* and whose semi-angle is 60° with

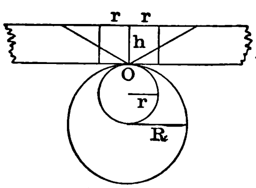


Fig. 4.

its vertex at O, will have an attraction at O equal to that of the cylinder, and to half that of the infinite plate.

As will appear later, the large sphere of radius $R = \frac{3}{2}h$ will balance the attraction of the infinite plate on a particle at the point of tangency, and the small sphere inscribed on its radius will balance the attraction of the cone or the cylinder shown in the figure. If the small sphere were to represent a cavity in the larger one, and either the cone or cylinder were to be removed from the infinite plate, the attractions on O of the remaining matter would balance.

27. *Spherical Shell.*

The attraction of a thin homogeneous shell on a particle within is zero.

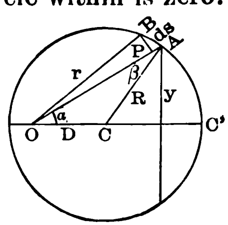


Fig. 5.

Let the particle be at any point O, through which point draw the diameter $O\ C$. Let $A\,B = d\,s$ be a short arc of a great circle, distant r from O. Revolve $A\ B$ around $O\ C$, keeping $\angle\, B\ O\ C = a$, constant. Let R be the radius of the sphere, and $\rho\ d\ r = \sigma$, the quantity of matter in astronomical units per unit area of the shell. The mass of the shell will then be

$$d\,m = 2\,\pi\,y\,d\,s\,\sigma.$$

The resultant attraction of the annulus on unit mass at O is along $O\ C$, and is

$$d\,A = \frac{d\,m}{r^2}\cos a = \frac{2\,\pi\,\sigma\,y\,d\,s}{r^2}\cos a.$$

We have also

$$y = r\sin a$$

$$ds = \frac{dr}{\cos B\,A\,P} = \frac{dr}{\sin\beta}$$

$$\therefore\ y\,d\,s = r\,d\,r\frac{\sin a}{\sin\beta} = r\,d\,r\,\frac{R}{D}$$

where $D = O\ C$.

The triangle $O\ A\ C$ gives

$$\cos a = \frac{r^2 + D^2 - R^2}{2\ r\ D} = \frac{1}{2D}\left(r + \frac{D^2 - R^2}{r}\right)$$

By substitution of these values in the equation for attraction we have by integrating in r over the surface of the sphere

$$A = \frac{a\,\sigma\,R}{D^2}\left[\int_{-(D-R)}^{D+R} dr + (D^2 - R^2)\int_{-(D-R)}^{D+R} \frac{dr}{r^2}\right] \ldots (13).$$

$$= \frac{\pi\sigma\,R}{D^2}\left[D + R - R + D + (D^2 - R^2)\left(\frac{1}{R-D} - \frac{1}{R+D}\right)\right] = 0$$

28. *Attraction on an external Point.*

If the attracted particle be external to the shell, the same discussion applies, but the limits of integration are $D + R$ and $D - R$. The attraction then becomes,

$$A = \frac{\pi\sigma\,R}{D^2}\left[D + R - D + R + (D^2 - R^2)\left(\frac{1}{D-R} - \frac{1}{D+R}\right)\right]$$

$$= \frac{4\,\pi\,R^2\,\sigma}{D^2} = \frac{M}{D^2} \ldots\ldots\ldots\ldots\ldots\ldots (14).$$

The attraction on an external point is the same as it would be were the entire mass of the shell at the centre.

29. *Attraction of a solid Sphere.*

In the last equation (14) if we replace σ by ρdR and integrate in R between limits o and R the attraction of a solid sphere is determined. We thus have,

$$A' = \frac{4}{3}\,\frac{\pi\rho\,R^3}{D^2} = \frac{M}{D^2}.$$

The M of this equation is the sum of the masses of

the shells, the attraction of any one of which is represented in (14).

30. *Attraction of a Spherical Shell Sector.*

Returning to (13) if we integrate in r between $D + R$ and r' we have

$$A = \frac{\pi \sigma R}{D^2}\left[D + R - r' + (D^2 - R^2)\left(\frac{1}{r'} - \frac{1}{D + R}\right)\right] \quad (15).$$

The condition that A shall be a maximum can be satisfied for an internal point, when $R > D$ and the condition is $\frac{dA}{dr'} = o$, or

$$r'^2 = R^2 - D^2$$

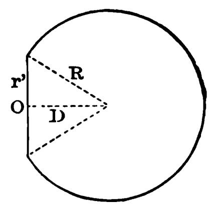


Fig. 6.

The integration must be extended to the plane which passes through O, at right angles to the diameter of symmetry. See Fig. 6.

The condition that the attraction of the spherical sector on O shall be $\frac{1}{n}$ of the attraction of the entire spherical shell, may be determined for an external point. We have by imposing this condition on (15) and (14) and reducing

$$r' = -R\left(\frac{2}{n} - 1\right) \pm \sqrt{(D^2 - R^2) + R^2\left(\frac{2}{n} - 1\right)^2}.$$

If, for example, $n = 2$

$$r'^2 = D^2 - R^2$$

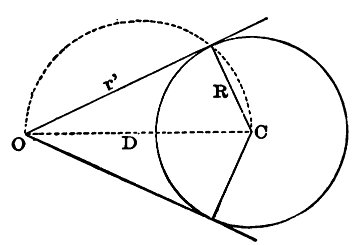


Fig. 7.

This shows that a cone, having its vertex at O, and to which the spherical shell is internally tangent, will at its circle of tangency divide the shell into two sectors which exert equal attractions on O. It will be observed that a sphere constructed on

$O\ C$ as a diameter, will intersect both shell and cone in the circle of tangency. The same sphere on $O\ C$ would divide any concentric shell having its centre at C into two parts, each having one-half the attraction on O exerted by the entire shell.

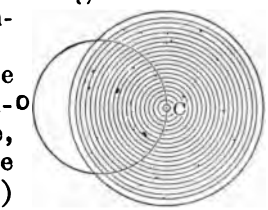


Fig. 8.

If the point O be at the surface of a solid sphere, the sphere constructed on its radius will, therefore, exert on O half the attraction of the entire solid sphere. (See Fig. 4.) The same considerations hold for a point O inside a solid sphere, since the shell external to O exerts no attraction on O or on points internal to O, and since O is at the surface of the internal spherical mass to which the attraction at O is alone due. (See Fig. 9.)

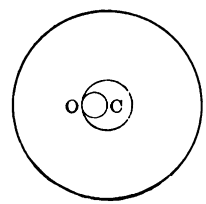


Fig. 9.

31. *Attraction within a Gravitating Mass.*

The attraction on a unit mass within a solid homogeneous sphere, at a distance R from the centre is

$$A = \frac{4}{3}\,\pi\,\rho\,\frac{R^3}{R^2} = \frac{4}{3}\,\pi\,\rho\,R.$$

The force varies directly as the distance from the centre.

In a continuous homogeneous medium of infinite extent, having, therefore, no free surface, no point in the medium can be considered distinctively as the centre. The forces would be balanced at every point, as they are at the centre of a limited mass.

CHAPTER II.

POTENTIAL.

32. *Potential due to a single mass.*

At any distance r from an attracting mass m, the attraction in dynes on a unit mass will be

$$f = \frac{m}{r^2} \text{..............................} (16).$$

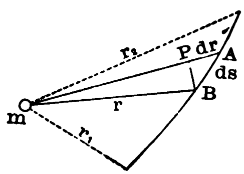


Fig. 10.

If the unit mass be carried from B to A over a distance $d\,s$, the work done will be $d\,s$ multiplied by the component of the force along $d\,s$ or

$$d\,w = \frac{m}{r^2} \cos a\, d\,s \text{ ....} (17),$$

where $a = \angle P\,A\,B$. Since $\cos a = \dfrac{dr}{ds}$

we have

$$dw = m\,\frac{dr}{r^2} \text{..................................} (18).$$

The path $B\,A$ in Fig. 10 is, therefore, to be considered as a frictionless inclined plane. Equations (17) and (18) assert that the work required to slide the unit mass from B to A on the inclined surface is the same as that required to lift it vertically along the lines of force *from P to A,* which is the height of the plane.

Integrating in r between r_1 and r_2 we have

$$w = m\int_{r_1}^{r_2}\frac{dr}{r^2} = \frac{m}{r_1} - \frac{m}{r_2} \text{ .... } (19).$$

This expression gives the work done in carrying unit mass along any path, against the attraction $\frac{m}{r^2}$, from any point distant r_1 to any other point distant r_2 from m. This work depends solely upon the initial and final distances r_1 and r_2, and not upon the geometrical form of the path. If the path lie wholly in the spherical surface whose radius is r_1, the work on every element of the path is zero. If the path be any other whatever which begins and ends in the same spherical surface, so that $r_1 = r_2$, the total work is zero.

It will be seen that the expression (19) is a difference between two like quantities.

If $r_2 = \infty$ then

$$w = \frac{m}{r_1} \text{ .... } (20).$$

This represents the work required to carry unit mass (3928 grammes) from a point at a distance r_1 from m to an infinite distance. It is then evident that $\frac{m}{r_2}$ is the work required to carry unit mass from a point at a distance r_2 from m to an infinite distance. The difference between these quantities, therefore, represents the work required to carry the unit mass from one point to the other.

The quantity $\frac{m}{r}$ is thus seen to be one of great importance, and it has received a special name. It is said to be the *potential* due to m at a distance r from the acting mass. On any spherical surface whose centre is

m, the potential is constant. As r increases the potential diminishes, and it becomes zero when $r = \infty$.

If m were a sphere of brass having 1,000 units of mass or 3,928,000 grammes, then at a distance of 1,000 centimetres or 10 metres, the potential would be unity. The work required to carry 3928 grammes from any point distant 1,000 cm. to an infinite distance would be one dyne-centimetre, or one erg. The force at the initial position would be 0.001 dyne. When the point determined by the distance r is within the acting mass m, the law of potential changes as the law of force has been shown to change.

33. *Magnitude of the Unit of Potential.*

The unit of potential must be directly proportional to the unit of electrical quantity, and inversely as the unit of length. (See Eq. 20, and section 11.) Hence its magnitude is

$$\frac{M^{\frac{1}{2}}\ L^{\frac{3}{2}}\ T^{-1}}{L} = M^{\frac{1}{2}}\, L^{\frac{1}{2}}\, T^{-1}.$$

It may also be defined as work per unit quantity or

$$\frac{M\ L^2\ T^{-2}}{M^{\frac{1}{2}}\, L^{\frac{3}{2}}\ T^{-1}} = M^{\frac{1}{2}}\, L^{\frac{1}{2}}\, T^{-1}.$$

34. *Strength of Field and Potential.*

The relation between these quantities can be illustrated graphically as in Fig. 11.

Calling V the potential, as is customary, the two equations are

$$V = \frac{m}{r}$$

$$f = \frac{m}{r^2}$$

The strength of field due to a mass m at a distance r is represented by the ordinate of the curve marked f. The work done over a distance dr is $fdr = \frac{m}{r^2}dr$. This element of work is represented by an element of area of the curve. The integral of fdr from r_1 to an infinite distance is represented by the area of the force curve out to an infinite distance and is equal to $\frac{m}{r_1}$. This quantity is the ordinate of the potential curve at r_1. The potential curve is an equilateral hyperbola. Its ordinate at any distance r, is equal to the area of the force curve from that point to $r = \infty$.

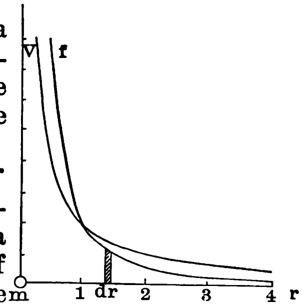


Fig. 11.

35. *Gravitation Potential.*

To determine the gravitation potential at the surface of the earth we are to find the work required to carry 3928 grammes from the earth's surface to an infinite distance.

The mass of the earth in grammes is 6.14×10^{27}. Its mass in astronomical units is

$$M = \frac{6.14 \times 10^{27}}{3928}.$$

The mean radius of the earth is 6.37×10^8 centimetres. Hence the potential at the earth's surface is by (20)

$$V = \frac{M}{R} = \frac{6.14 \times 10^{27}}{3928 \times 6.37 \times 10^3}$$

$$= 2.454 \times 10^{15}.$$

This is the work in ergs that must be applied to each unit of mass to carry it from the earth's surface to an

infinite distance. Assuming $g = 981$ this work would be in centimetre-grammes at the earth's surface

$$\frac{2.454 \times 10^{15}}{981} = 2.501 \times 10^{12}$$

or 2.501×10^7 kilogramme-metres or 25010 tonne-metres.

This work would, therefore, be the same as would be required to lift 25,010 tonnes of 1000 kilogrammes through a height of one metre. A day's work for a laborer is 100,000 kilogramme-metres. It would require the labor of a man for 250 days to do the work computed above. Neglecting the protecting effect of the earth's atmosphere, the damage which a meteorite having unit mass could inflict on the earth if it had a velocity due to an infinite fall under the earth's attraction is represented by 2.501×10^7 kilogramme-metres. This velocity is determined by the equation

$$\frac{3928\ v^2}{2} = 2.454 \times 10^{15}$$

$\therefore v = 1.117 \times 10^6$ centimetres per second or 11.17 kilometres per second.

This is a little less than seven miles per second.

36. *Potential due to any system.*

If any system of bodies $m_1\ m_2\ m_3$ etc., act on a unit mass each body will continually exert the same force upon it as if it were acting alone. In order to get the resultant force in the field we should construct the polygon of forces taking account of the direction and intensity of each force. But potential is not a directed quantity. The resulting potential will be the algebraic sum of the potentials due to the several masses, or

$$v = \Sigma \frac{m}{r} \quad \dots\dots\dots\dots\dots\dots\dots\dots (21).$$

If the acting mass is in the form of a thin shell, the *potential at any* point due to the shell must be deter-

mined by integration over its surface. If σ be the quantity of matter per unit area, of the surface, then dS being any element of area and r being the distance of the element of area from the testing unit, the potential due to the shell will be

$$V = \int \frac{\sigma d\,S}{r} \text{ ....(22).}$$

This integration can only be effected when the geometrical form of the shell surface is known and can be expressed by an additional equation.

37. *To find the potential at any point within a thin homogeneous shell, bounded by concentric spherical surfaces.*

The mass of the zone generated by revolving the arc $A\ B$ having a length ds about the diameter $O\ C$ is

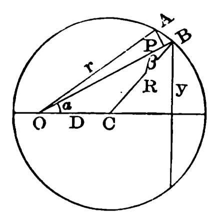


Fig 12.

$$d\,m = 2\,\pi\,y\,d\,s\,\sigma$$

where $y = r \sin a$

$$ds = \frac{dr}{\cos P\,B\,A} = \frac{dr}{\sin\beta}$$

since $C\ B$ is at R ∟ to $A\ B$.

$$\therefore\ yds. = rdr\frac{\sin a}{\sin\beta} \doteq rdr\,\frac{R}{D}$$

$$\text{and } dm = 2\,\pi\,\sigma\,\frac{R}{D}\,r\,dr$$

The potential at O due to the zone having a mass dm, every part of which is equidistant from O is

$$d\,V = \frac{dm}{r} = 2\,\pi\,\sigma\,\frac{R}{D}\,dr.$$

Since potential is not a directed quantity this potential is not resolved along the axis of symmetry as was done in determining the force within a spherical shell. (See section 27.)

The potential due to the entire spherical shell is then

$$V = 2\pi\sigma\frac{R}{D}\int_{-(D-R)}^{D+R} dr$$

$$= 2\pi\sigma\frac{R}{D}\Big[D+R+D-R\Big]$$

$$= 4\pi\sigma R = 4\pi\sigma\frac{R^2}{R} = \frac{M}{R} \quad \text{.............(23).}$$

The potential at any point O within the shell is the same as the potential at the centre, and is, therefore, constant.

This result follows from the fact proved in section 27, that the force within the shell is zero. It, therefore, requires no work to carry unit mass from one point to any other point within the shell. Therefore there is no difference of potential between such points, or the potential is constant. Since the center is equidistant from each element of the shell, it is evident that the potential at that point must be

$$V = \frac{\int dm}{R} = \frac{M}{R}.$$

The potential at the shell surface is, therefore, also $\frac{M}{R}$, which is the same as it would be at such points, if the mass of the shell were concentrated at the centre.

If the point O be outside of the shell, the same considerations hold, but the limits of integration are $D + R$ and $D - R$. The potential due to the shell at an external point is

$$V = 2\pi\sigma\frac{R}{D}(D + R - D + R),$$

$$= 4\pi\sigma\frac{R^2}{D} = \frac{M}{D}. \quad \text{.......... ....(24),}$$

which is the same as it would be if the mass of the shell

CHAPTER I.

ATTRACTION.

20. *Law of Attraction.*

It has been established by experiment that the attraction between two small spherical masses, separated by a distance r, is

$$A = K\frac{m\,m'}{r^2} \text{..........................} (1).$$

If the two bodies approach each other under this attraction the accelerations of the bodies being a and a' respectively, then the force A is also

$$A = K' m\,a = K' m'\,a' \text{................} (2).$$

In (1) if $m = m' =$ one gramme, and $r =$ one centimetre, then

$$A = K.$$

The constant K may be defined as the force with which unit mass will attract an equal mass distant one centimetre.

In (2) if $m = 1$ and $a = 1$

$$A = K'.$$

Therefore, K' may be defined as the [illegible] give unit acceleration to unit mass. [illegible] has been universally adopted as the unit [illegible] the C. G. S. system this unit is called [illegible] unit (2) becomes

$$A = m\,a = [illegible]$$

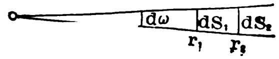
timetre being numerically equal to the strength of the field. The lines of force proceeding from a thin hollow spherical shell, would be radial lines, but they terminate in the shell, since the strength of the field within is zero. In a solid sphere the lines within must thin out as we approach the centre, a certain number terminating in each concentric shell. The number of lines reaching the centre of the sphere will be zero.

At a distance r from a mass m, the number of lines to the square centimetre is $\frac{m}{r^2}$. The area of a sphere of radius r is $4\pi r^2$. Therefore the total number of lines proceeding from a mass m and crossing the spherical surface of radius r is $\frac{m}{r^2} \times 4\pi r^2 = 4\pi m$. This quantity is independent of r and depends only on the mass from which the lines proceed. The number of lines proceeding from unit mass is, therefore, 4π.

If consecutive surfaces of equal potential be drawn at right angles to the lines of force, these surfaces will lie closer together the greater the strength of the field. For the surfaces are so drawn that the work required to carry unit mass between consecutive surfaces is one erg. Where the strength of the field is great, the distance over which the force must be exerted in order to do a unit of work will be correspondingly small.

It will be understood that lines of force cross equipotential surfaces at right angles. If this were not so, there would be a component of force along the equipotential surface. It would then require work to carry unit mass along the surface against the component. But this is contrary to the assumption that the surface is of constant potential.

40. *Tubes of Force.* Any cone having its vertex in a mass m and bounded by lines of force, is called a tube of force.



If the section of a tube at any distance r from a particle

m is $d\ S$, then the solid angle of the cone is (section 19)

$$d\ \omega = \frac{dS_1}{r_1^2} = \frac{dS_2}{r_2^2}.$$

The strength of the field due to m being denoted by f we also have

$$m = f_1\, r_1^2 = f_2\, r_2^2,$$

Hence

$$m\, d\ \omega = f_1\ d\ S_1 = f_2\ d\ S_2 \text{..............} (25).$$

This product $f\, ds$ is, therefore, constant throughout the tube. The strength of field at a distance unity from m, is m. The section of the tube at that point is $d\ \omega$. The three expressions in (25) are, therefore, expressions for the number of lines of force within the tube at various points. If these expressions are integrated around the mass m at a constant distance r, the integral of the first is $m \int d\ \omega = 4\ \pi\ m$, where $4\ \pi$ is the area of a sphere of unit radius, or the solid angle around a point. The other terms of the equation become on integration around the mass m at constant distance r

$$f \int dS = \frac{m}{r^2}\, 4\ \pi\ r^2 = 4\ \pi\ m.$$

If the section dS' of the tube be oblique, making an angle a with the right section, then

$$d\ S = d\ S' \cos a.$$

Calling the component of f at right angles to the oblique section f' then

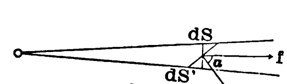


Fig. 14.

$$f \cos a = f'.$$

Multiplying these equations together,

$$f\, d\ S = f'\ d\ S'$$

41. *Flow of Force.* The relation between strength of field and section of a tube is similar to the relation between velocity of flow of an incompressible fluid in a tube bounded by stream-lines, and section of the tube of flow. The product of velocity and section represents the discharge per second. This discharge is constant throughout the tube or $v\,d\,S = v'\,d\,S'$. Where the section is small, the velocity is great. The component of velocity across an oblique section into the area of that section gives the same discharge. All of these relations have just been shown to exist between f and $d\,S$. Where the lines of force crowd together, the force on a unit mass is great and the section of a given tube diminishes, so that $f\,dS$ is unchanged. If the lines of force are parallel, the force is constant. This is the case in front of an infinite attracting plane. It is the case very near to any electrified surface.

42. *The lines of force radiating from a mass m, between two infinite parallel planes, all cross one or the other of these planes.*

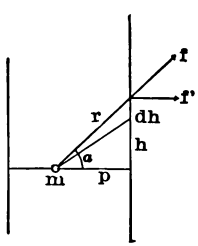


Fig. 15.

The force on the plane at any distance r from m is $f = \frac{m}{r^2}\cdot$ We have just found (section 40) that we must compute the component of force at right angles to the surface across which the flow is to be determined. If p is the perpendicular distance from m to the plane, and a the angle between r and p as in Fig. 15, then the component of f at R $\perp$ to the plane is $f' = f\,cos\,a = \frac{mp}{r^3}\cdot$

Let h be the distance on the plane from the foot of p to the point determined by r. Then f and f' will be constant over an annulus having a radial width $d\,h$ and a radius h, each point of the annulus being distant r from m. The area of this annulus is $2\pi h\ d\ h$. Since $r^2 = h^2$

$+ p^2$ we have $h\ d\ h = r\ d\ r$, p being constant. Therefore the area of the annulus is

$$d\ S = 2\pi r\ d\ r.$$

The force over this annulus at $R \perp$ to its surface is

$$f' = \frac{m\ p}{r^3}$$

$$\therefore f'\ d\ s = 2\pi m\ \ p\ \frac{dr}{r^2}.$$

The total number of lines crossing the plane from m is, therefore,

$$N = \int f'\ d\ s = 2\ \pi\ m\ p \int_p^\infty \frac{dr}{r^2}$$

$$= 2\ \pi\ m\ \ p \left(\frac{1}{p} - \frac{1}{\infty} \right) = 2\pi m.$$

This integral is, therefore, independent of p, the distance of the particle from the plane, and in value it is equal to half of the number of lines which proceed from m. The other half of the $4\pi m$ lines will, therefore, cross the other plane. All the lines proceeding from m which depart in the slightest from a plane parallel to the two planes, will cross one or the other at a finite distance from m. Since at an infinite distance the force is zero, the number of lines between the planes at an infinite distance from m is zero.

43. *Mass within a Spherical Surface.*

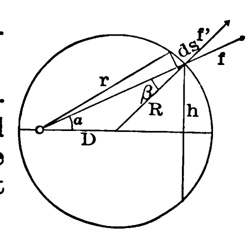


Fig. 16.

If a mass m be placed eccentrically within a spherical surface, and distant D from the centre, the force at any point on the sphere distant r is $\frac{m}{r^2}$. This will be the force on

a zone having a radius h, and a width $d\,s$. The component of the force along a radius R of the sphere is

$$f' = \frac{m}{r^2} \cos \beta$$

where β is the angle between r and R.
Since

$$\cos \beta = \frac{r^2 + R^2 - D^2}{2rR}$$

we have by separating this fraction and substituting in the previous equation,

$$f' = \frac{m}{2Rr} + \frac{m\,(R^2 - D^2)}{2Rr^3}.$$

The area of the zone is

$$d\,S = 2\pi h\,d\,s$$

$$\text{where } h = r \sin a, \quad d\,s = \frac{dr}{\sin \beta}$$

$$\therefore d\,S = 2\,\pi\,\frac{\sin a}{\sin \beta}\,r\,d\,r$$

$$= 2\,\pi\,\frac{R}{D}\,r\,d\,r$$

and

$$f'\,d\,S = \frac{\pi\,m}{D}\,dr + \pi m\,\frac{R^2 - D^2}{D}\frac{dr}{r^2}.$$

Therefore the flow of force across this spherical surface from m will be

$$N = \int f'\,d\,s = \frac{\pi\,m}{D}\int_{-(D-R)}^{D+R} dr + \pi\,m\,\frac{R^2 - D^2}{D}\int_{-(D-R)}^{D+R}\frac{d\,r}{r^2}$$

$$= \frac{\pi\,m}{D}(2\,D + 2\,D) = 4\,\pi\,m.$$

CHAPTER III.

ELECTROSTATICS.

44. The theorems which have preceded apply directly to the action upon each other of electrical masses in equilibrium. The unit of electrical quantity has already been defined in sections 10–11. In electrostatics we must, however, consider the action of both positive and negative masses. Gravitating matter always acts by attraction. Electrical masses repel each other when the masses have like signs, whether positive or negative, while when the charges have unlike signs they attract each other.

When equal charges on two equal spheres attract each other, the spheres both become unelectrified when they are put in metallic contact. The sum of the two charges is zero. When two equal charges thus repel each other, and the two spheres are put in contact, the charges remain unaffected.

The astronomical equation for attraction is found to represent the action between such electrical masses. We have

$$A = \frac{m\,m'}{r^2} \qquad (26).$$

The sign of A will be positive if both m and m' are positive or if both are negative. A repulsion, therefore, is represented as a positive force. If either m or m' be negative and the other positive, the sign of A will be negative, and this will represent an attraction.

The potential due to a positive charge is assumed to be positive in sign, and that due to negative charges is negative. The potential due to any mass m of positive electricity at a distance r is

$$V = \frac{m}{r} \qquad (27).$$

The testing unit which is supposed to be placed at the point distant r is always a positive unit. Accordingly the work required to bring a plus unit from an infinite distance where the potential is zero into a region of positive potential, will be positive in sign. To bring the plus unit from an infinite distance into a region of negative potential will require negative work, to be done on the unit by the external source of energy, since the electrical forces to which the field is due attract the testing unit. In this case it will require positive work to carry the testing unit out of the field. Lines of force are said to originate in positive masses and terminate in negative masses. The plus direction of a line of force is assumed to be the direction in which the plus testing unit is urged by the electrical forces. If by hydraulic analogy we speak of the flow of force through the tubes of force or along the stream lines, we assume the flow to be outward from positive charges, and to converge upon negative masses. Such expressions must be regarded only as convenient fictions which mathematical similarities justify us in using, just as for convenience we do not hesitate to speak of the rising and the setting of the sun.

45. *Field due to two equal masses of unlike sign.*

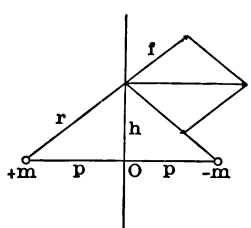


Fig. 17.

Let m and $-m$ represent two equal masses of unlike sign, the distance between them being $2p$. If a very small sphere having its centre at m surround the plus particle the force over this sphere is due to m, the force due to $-m$ being inappreciable by comparison. This force is $\frac{m}{r^2}$ and the number of lines diverging from m and crossing the spherical surface will be $\frac{m}{r^2} \times 4\pi r^2 = 4\pi m$. A similar sphere around $-m$ would be traversed by an equal *number of negative* lines which converge upon $-m$. If

we consider these two masses separately, the lines of force due to each will be a system of radial and equally distributed lines in all directions. When we consider the two particles as forming a system, the resulting lines of force due to the two bodies must be determined by combining the separate action of the two bodies at every point in the field, by means of the parallelogram of forces.

Let us first determine the resultant force on the plane midway between the two particles and at right angles to the line joining them. (See Fig. 17.)

The repulsive force on a + unit due to m at any point on the plane distant r, would be $\frac{m}{r^2}$, and directed away from m. The attraction due to $-m$ at the same point would be $\frac{-m}{r^2}$ and would be directed towards $-m$. These two forces are equal and make equal angles with the normal to the plane at the point. Their resultant is, therefore, normal to the plane.

The angle α between each force and the resultant, is the same as that between the lines p and r. Hence the resultant force on the + unit is

$$f = 2\frac{m}{r^2}\cos\alpha = 2\frac{mp}{r^3} \quad \text{.......................} (28).$$

The number of lines per square centimetre in the field of m and $-m$ at any point determined by r on this plane is, therefore, determined by (28). The force on a + unit varies inversely as the cube of the distance, and the force is parallel to the line joining the two masses.

To determine the total number of lines due to the system, which cross this plane, we must integrate $f\,d\,S$ over the plane. On an annulus every part of which is distant r from the two masses, whose radius is h, and whose radial width is dh, f will be constant, and its value is given in (28). The area of the annulus is

$$\begin{aligned} d\,s &= 2\pi h\,d\,h \\ &= 2\pi r\,d\,r \end{aligned}$$

since $h^2 + p^2 = r^2$, p being constant

line passing through the acting masses at points where

$$r_1 + r'_1 = 2\,a \text{ and}$$
$$r_2 - r'_2 = 2\,a.$$

We may refer the circle to rectangular co-ordinates with the origin at the point midway between the particles. We have

$$r^2 = y^2 + (a + x)^2$$

$$r'^2_1 = y^2 + (a - x)^2 = \left(\frac{m'}{m}\right) r^2$$

Eliminating r —

$$y^2 + (a - x)^2 = \frac{m'}{m}\Big(y^2 + (a + x)^2\Big)$$

$$\text{or } y^2 + \left(x - a\,\frac{m + m'}{m - m'}\right)^2 = a^2\,\frac{4\,m\,m'}{(m - m')^2}$$

If $y = o$

$$x = a\,\frac{m + m'}{m - m'} \pm \frac{2a}{m - m'}\sqrt{m\,m'} \quad \text{.............(48).}$$

The first term of the second member is the abscissa of the centre, and the second term is the radius of the circle. As m' approaches m in numerical value, both terms of this expression become very large, and approach each other and $\frac{2\,a\,m}{m - m'}$ in value. In the limit when $m = m'$ the distance to the centre becomes infinite, and the term $\frac{2a}{m - m'}\sqrt{m\,m'}$ representing the radius has an equal value. Hence one point of intersection is then at the origin, and the infinite plane which bisects the line $m\,m'$ at $R\llcorner$, is then a portion of the sphere of zero potential.

If $a = 10$; $m = 10.001$ and $m' = 10$, the distance to the cen[illegible] sphere of zero potential would be 200,010

and the ± term representing the radius would be 200,-009.8, the intersections being distant from the origin 400,019.8 cm. or 4000 metres and 0.2 cm.

The curve for any other value of V is easily constructed from the equation, and the intersections with the axes can be readily obtained from either the polar equation (46) or from equation (48) when the distance between the particles is known.

57. *To determine the flow of force across the sphere of zero potential, enclosing the smaller mass* — m'.

Let D and D' be the distances of m and $-m'$ from the centre (Fig. 25). Let β and a be the angles made with the radius drawn to any point of the sphere by the lines r and r' joining the acting masses with the same point, the rectangular co-ordinates of the point referred to the centre C being x and y.

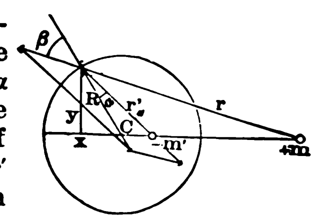


Fig. 25.

Then the resultant force due to m and $-m'$ at the surface of zero potential is

$$f = \frac{m'}{r'^2} \cos a - \frac{m}{r^2} \cos \beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(49),$$

this force being directed radially towards the centre of the sphere.

Further —

$$\cos a = \frac{r'^2 + R^2 - D'^2}{2r'R}\,;\ \cos \beta = \frac{r^2 + R^2 - D^2}{2rR} \ldots.(50).$$

Over a zone of the sphere whose width is ds, and which is everywhere equidistant from either m or $-m'$ this force will be constant. The area of this zone is

$$dS = 2\cdot$$

where

$$y = r \sin \omega = r' \sin \omega'$$

$$ds = \frac{dr}{\sin \beta} = \frac{dr'}{\sin \alpha}$$

and

$$\therefore yds = \frac{\sin \omega}{\sin \beta} rdr = \frac{\sin \omega'}{\sin \alpha} r' dr'$$

$$dS = 2\pi \frac{R}{D} rdr = 2\pi \frac{R}{D'} r' dr'$$

Substituting (50) in (49), and multiplying the resulting equation by dS after separating terms in (50)

$$\int fds = \frac{\pi m'}{D'} \int_{-(D'-R)}^{D'+R} dr' + \frac{\pi m' (R^2 - D^2)}{D'} \int_{-(D'-R)}^{D'+R} \frac{dr'}{r'^2}$$

$$- \frac{\pi m}{D} \int_{D-R}^{D+R} dr + \frac{\pi m (D^2 - R^2)}{D} \int_{D-R}^{D+R} \frac{dr'}{r'^2}$$

$$= 4 \pi m'$$

58. *Equilibrium of the charges m and — m' on equipotential surfaces of the system.*

It is evident that lines of force cut the equipotential surfaces of the system at right angles. If this were not so there would be a component of force along the surface, and it would require work to move the testing unit along the surface against that component. This is contrary to the supposition that the surface is one of constant potential.

We have seen that tubes of force originating in a charge m in equilibrium on a small sphere at A, (Fig. 26) *and terminating* in the spherical charge — m' at B have *equal charges upon* their ends. It therefore follows that

the entire charge on sphere *m*, may be transferred to any equipotential surface surrounding *m* without changing

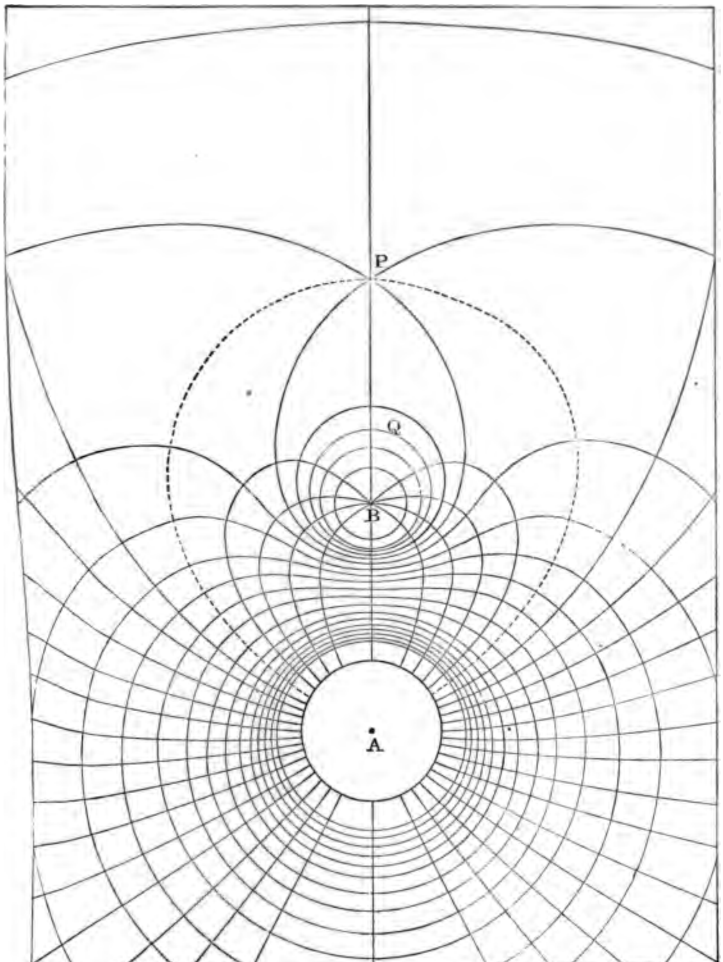


Fig. 26.

the field external to that surface. The amount within any tube *of force is to remain* the same as before. The

electricity is to be projected along the lines of force from one surface to the other. The potential within the newly charged surface will then be constant, and will be the same as that of the newly charged surface before the change. The area of this surface being greater than that of the original sphere, the density of charge will be less. Therefore the force f at the surface will be less than it was at the former surface of charge, but the intregal $\int f ds$ will remain unchanged, being $4\pi m$. If the equipotential surface were to become conducting, the charge thus distributed would be in equilibrium upon it. The portion of the charge m, which is within the critical surface, namely $+m'$, might be similarly spread over any equipotential surface enclosing $-m'$ at B and the field within would remain unchanged. If this were the surface of zero potential, marked Q, its potential in the presence of $-m'$ would still be zero, and the potential of all external points due to the two charges $-m'$ and $+m'$ would be zero. The distribution on the sphere of zero potential would then be the same as it would be if the sphere were conducting, and were to be grounded in presence of the charge $-m'$ within.

Returning now to the original condition of m and $-m'$, if we were to transfer the charge $-m'$ at B to the sphere of zero potential as before described, the external field due to $+m$ and $-m'$ would remain unchanged. The potential on and within the sphere of zero potential would be zero, and the distribution on that sphere would be the same as it would be if it were a conducting sphere, grounded in the presence of the external charge m.

It thus appears that if this sphere of zero potential due to the charges m and $-m'$ were a conductor, and grounded in presence of the external particle $+m$ or the internal particle $-m'$, induced charges would be produced which in the two cases would be identical in distribution, and opposite in sign. If the sphere were to be grounded in presence of both particles, its potential would remain zero, and the distributions on its outer and

inner surfaces would remain unchanged. We proceed to determine the law of density of this induced charge.

59. *Required the density of the induced charge at any point on a conducting sphere due to the inductive action of either an internal or an external particle.*

By Eq. 48, the distance from $-m'$ to the centre of the sphere is

$$D' = a\,\frac{m+m'}{m-m'} - a = \frac{2\,am'}{m-m'} \quad \text{............(52).}$$

The radius of the sphere is by the same equation.

$$R = 2a\,\frac{\sqrt{m\,m'}}{m-m'} \quad \text{....................(53).}$$

Eliminating m by (52) and (53)

$$D'\,(2a + D') = R^2,$$

or

$$D'\,D = R^2 \quad \text{..............................(54),}$$

or the radius of the sphere of zero potential due to two particles m and $-m'$, is a mean proportional between the distances of the particles from the centre. The position of these particles is at conjugate points of the sphere. Hence with a given sphere, if either D or D' be given the other can be computed.

The condition which will determine the hypothetical mass which, placed at any point within or without a sphere, in presence of a given mass at a conjugate point, will reduce the sphere to zero potential is

$$\frac{m}{r} = \frac{m}{r'} \quad \text{...(47).}$$

At the points of intersection of the sphere by the line through m, m' we have for the internal intersection,

$$r' = R - D' \qquad r = D - R$$

and for the external intersection

$$r' = D' + R \qquad r = D + R$$

from which

$$\frac{m}{m'} = \frac{D - R}{R - D'} = \frac{D + R}{D' + R} \quad \text{.....................(55).}$$

Eliminating D from either of these expressions (55) by means of (54)

$$m = \frac{R}{D'}\, m' \quad \text{.....................................(56).}$$

Eliminating D' we have

$$m' = \frac{R}{D}\, m \quad \text{..(57).}$$

If we have any internal mass $-m'$ at any distance D' from the centre of a sphere of radius R, then (56) gives the mass $+m$ that must be placed at a conjugate point, determined by (54) in order to produce zero potential on the sphere. Equation (57) applies in a similar manner to an external mass m. This equation is instructive, in showing that if $-m'$ were placed on the sphere of zero potential in presence of $+m$ at a distance D from the centre, the potential at the centre would be zero whatever might be the law of distribution of $-m'$ over the sphere.

The force at the surface of zero potential is given by

(49) and (50). Eliminating D, m and r by means of (54), (56) and (47) we have

$$f = \frac{R^2 - D'^2}{R} \frac{m'}{r'^3} \quad \text{.......................(58).}$$

This is the force on a plus unit at the surface of zero potential due to $+ m$ and $- m'$, or at, but inside the surface, due to the charge $- m'$ and the positive induced charge on the sphere, if it be a conducting surface grounded in presence of the internal particle, $- m'$.

In order to determine the density of distribution of the induced charge we must consider certain experimental results.

1. Suppose the sphere of Fig. 25 to be conducting and grounded, and let $- m'$ be any mass at any internal point. The force just outside of the internal spherical surface is zero. The attraction due to $- m'$ is exactly balanced by the repulsion of $+ m'$ induced upon the sphere. Hence, at a point just outside this distribution, of quantity $+ m'$, it acts as if it were at the point occupied by $- m'$.

2. If a mass $- m$ determined by Eq. (56) be placed at a conjugate point, we have shown that it would produce the same distribution on the sphere. Experiment shows that the force within the sphere, due to $- m$ at the conjugate point and the distribution $+ m'$ on the sphere is zero. The sphere acts as a perfect screen. The forces balance. Therefore, at a point inside the surface the distribution acts as $m = \frac{R}{D'} m' = \frac{D}{R} m'$ units would act at the conjugate external point.

At a point infinitely near the attracting surface, on either side, its attraction on a unit particle will be the same as that of a plate, or $2 \pi \sigma$. See Eq. (10). Since this attraction balances that of the inducing mass on the side of the surface where they are opposed, the force on the opposite side where they act together must be

$$f = + 4 \pi \sigma \quad \text{.............................(59).}$$

$$\text{Since } a^2 y^2 + b^2 x^2 + a^2 x^2 = a^2 b^2 + a^2 x^2$$

$$\therefore\ y^2 + x^2 = b^2 + e^2 x^2,$$

where e is the eccentricity of the ellipse of which a is the major axis.

Hence

$$\frac{1}{C_p} = \frac{1}{2a} \int_{-a}^{+a} \frac{dx}{\sqrt{b^2 + e^2 x^2}}$$

$$= \frac{1}{2\,ae} \Big[lg\,(ex + \sqrt{b^2 + e^2 x^2}) \Big]_{-a}^{+a}$$

$$= \frac{1}{2\,ae} lg\, \frac{1+e}{1-e} \text{ ....(89).}$$

This gives the capacity of a prolate spheroid. This expression can be developed into a series by means of McLaurin's formula and the value of C_p determined when $e = o$ or when the ellipsoid becomes a sphere. The value is $C_p = a$.

For any other value.

$$C_p = \frac{2ae\ M}{log_{10} \frac{1+e}{1-e}}$$

where $M = 0.4343$ or the modulus of common logarithms. If $e = 0.5$ or $b = a\sqrt{0.75} = 0.866\ a$ then

$$C_p = \frac{0.4343\ a}{0.4771} = 0.907\ a.$$

where a is the length of the major axis in centimetres.

If a is large compared with b it will be observed from the equations that the density approaches uniformity on the main body of the surface, but becomes very

large at its pointed ends. The element of charge on a zone whose height is dx is

$$dQ = \sigma dS = \frac{Q}{4\,\pi\,ab^2}\,pdS$$

or by equations leading to (88)

$$dQ = \frac{Q}{2a}\,dx, \text{ or}$$

$$\frac{dQ}{dx} = \frac{Q}{2a}.$$

The charge per unit of length along a is, therefore, constant, $= \frac{Q}{2a}$, or zones of equal height have equal charges. Since the surface per unit length may become very small at the pointed ends, the density will become correspondingly great.

71. *Oblate Spheroid.* If $a > b$, the axis b being the axis of rotation,

$$\text{Since } a^2y^2 + b^2x^2 + b^2y^2 = a^2b^2 + b^2y^2$$

$$x^2 + y^2 = a^2 - \frac{a^2e^2}{b^2}\,y^2$$

and (88) becomes

$$\frac{1}{C_o} = \frac{1}{2b}\int_{-b}^{+b}\frac{dy}{\sqrt{a^2 - \frac{a^2e^2}{b^2}\,y^2}}$$

$$= \frac{1}{2ae}\left[\text{arc-sin}\,\frac{ey}{b}\right]_{-b}^{+b} = \frac{\text{arc-sin}\,e}{ae}$$

$$\text{or } C_o = \frac{ae}{\text{arc-sin}\,e} \text{ ....(90).}$$

As e becomes small this value approaches a, so that this formula also gives the proper value for the capacity of a sphere.

When $b = o$ or $e = 1$, this ellipsoid becomes a circular plate, and

$$C = \frac{2a}{\pi} \quad \text{.......................................} (91).$$

When $e = 0.5$, then *arc-sin* $e = \frac{\pi}{6}$ and by (90) the capacity of the oblate spheriod having $b = 0.866\ a$ is

$$C_o = 0.954\ a.$$

72. *Elliptical Plate.*

The density at any point, x, y, z on the general ellipsoid is given in (85).

From the equation of the ellipsoid

$$\frac{z^2}{c^4} = \frac{1}{c^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right).$$

Substituting this value in (84) and leaving off the accents,

$$\frac{p}{c} = \frac{1}{\sqrt{c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right) + \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}}.$$

As c approaches zero $\frac{p}{c}$ approaches the limit,

$$\left(\frac{p}{c}\right)_{c=o} = \frac{1}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}$$

Therefore the ·density on one side of an elliptical e is

$$\sigma = \frac{Q}{4\pi\ ab}\ \frac{1}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}$$

For a circular plate $a = b$, and $x^2 + y^2 = r^2$ where r is any distance from the centre of the plate, of which a is the radius. Hence the density at any point on a circular plate becomes

$$\sigma = \frac{Q}{4\pi a\sqrt{a^2 - r^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (92).$$

The density at the centre of the plate on one side is $\frac{Q}{4\pi a^2} = \sigma_0$. The quantity on one side of the plate being $\frac{Q}{2}$ and the area of the plate being πa^2, the average density over the plate is $\frac{Q}{2\pi a^2}$ which is twice the density at the centre.

The plate has the average destiny at a point whose distance r is $r_a = a\sqrt{\frac{3}{4}} = 0.87\ a$. The density at the edge of the plate where $r = a$ is infinite. The density at any point in terms of the density at the centre is

$$\sigma = \frac{a\,\sigma_0}{\sqrt{a^2 - r^2}} \quad \ldots\ldots\ \ldots\ldots\ldots (93).$$

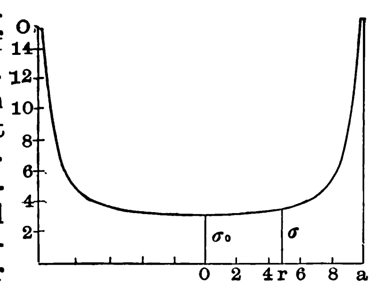


Fig. 32. Density on a Circular Plate.

The curve showing the density along a diameter of the plate is given in Fig. 32. If the curve shown in Fig 32 be revolved about the vertical line at the centre, representing the density at the centre, it will generate a surface, having right circular cylinder formed on the plate as a base, as an asymptote. The volume enclosed between plate, cylinder and surface of revolution will represent the charge $\frac{Q}{2}$ on one side of the

containing these circles divide all space around the masses $+m$ and $-m$ into ten equal conductors, each of which has a resistance of $10 \times \frac{1}{2\pi a} = \frac{5}{\pi a}$. The general form of these lines is represented in Fig. 38.

80. *On Dielectrics.*

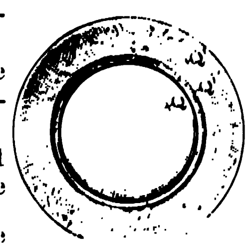
A region filled with air or other gases offers practically the same resistance to the flow of force as a vacuum. Cavendish investigated the condition of different media subjected to electric stresses, about a century ago. The fact that a medium can be ruptured when these stresses become large, and that there is a tendency of positive electricity to flow along lines of force towards lower potentials, or of negative electricity to flow in the opposite direction, is an element of the problem. This is analogous to a flow of heat along lines at every point at right angles to isothermal surfaces from regions of higher to regions of lower temperature, or to a flow of cold in an opposite direction.

The result of these electric stresses developed at every point in the medium, is the slow accumulation of surface charges by what is called inductive action. In a perfect conductor, or even in an ordinary metal conductor, these surface charges accumulate instantly, or so quickly, that they may be considered as instantaneous. The result is that these charges, taken in connection with the inducing charge, produce a constant potential in a piece of metal placed in a field of force. The lines of force in the metal then disappear. The force within becomes zero. In a dielectric like shellac, the surface charges accumulate very slowly, and are much less in density than they would be on a conductor. They are insufficient to produce constant potential within the body of the dielectric, but they do modify that potential. The force within is not made zero, but it is diminished. These fictive layers, as they are called, cannot be apprebly removed by making contracts with metal conduct-

ors, since they are distributed over a poor conductor on which the charge cannot travel.

If while the dielectric is under stress, a flame be played over its surface, each point of the surface will be discharged, and if the body be then removed from the inducing field, unbalanced internal stresses then appear, which are opposite in direction from those which formerly opposed external inductive action, and the surfaces will then develop charges opposite in sign from those formerly removed.

81. *Spherical Conductor, surrounded by a concentric shell of shellac or other dielectric.*



$r_0\ r_1\ r_2$

Fig. 89.

Let r_0 = radius of the sphere having a charge $+Q$.

Let r_1 and r_2 be the radii of the inner and outer surfaces of the concentric shell.

The force F_1 at the distance r_1 but inside the surface, due to the charge $+Q$ is $\frac{Q}{r_1^2}$. The force due to the fictive layers on the shell surfaces will be zero, whatever their amount, since the point is within them.

If σ_1 be the density of the fictive layer, then the force F_1' at the surface, but external to it, and, therefore, within the body of the shell infinitely near the former position, will be determined by the equation

$$F_1 - F_1' = 4\pi\sigma_1 \qquad (104.)$$

We may also write

$$\frac{F_1}{F_1'} = \mu \qquad (105)$$

and proceed to investigate the nature of the quantity μ. It is apparent at once that if the shell were a c

CHAPTER V.

ENERGY.

89. *Energy of an electrification.*

A body having a charge Q and a capacity C, has a potential

$$V = \frac{Q}{C}.$$

If the charge be increased by dQ, the work done in adding this charge is

$$dW = VdQ = \frac{Q}{C}\,dQ.$$

The total work done in charging the body from zero potential is

$$W = \frac{1}{C}\int_0^Q Q\,dQ = \frac{1}{2}\frac{Q^2}{C}$$

$$= \frac{1}{2}QV = \frac{1}{2}CV^2 \quad \text{..................(126).}$$

In the operation of charging, the average work per unit added is $\frac{1}{2}V$. This multiplied by the total quantity added, gives $\frac{1}{2}\,Q\,V$ as above.

90. *Energy of a charged sphere.*

The force with which any element of surface dS, having a charge $\sigma\, dS$, is apparently repelled outward by the rest of the electrification is

Fig. 43.

$$dp = 2\,\pi\sigma \times \sigma dS = 2\,\pi\sigma^2 dS$$

But $d\,S = r^2\, d\omega$

where $d\omega$ is the solid angle subtened by dS. Also,

$$\sigma = \frac{Q}{4\pi r^2}$$

$$\therefore dp = \frac{Q^2}{8\pi r^2}\, d\omega.$$

If the sphere be collapsed by external pressure, the radius being shortened dr, the work applied to the element against the opposing repulsion is

$$d^2 W = dp\; dr = \frac{Q^2}{8\pi}\;\frac{dr}{r^2}\, d\omega.$$

The total work required to shorten the radius of the sphere from r'' to r' will be found by integrating in ω over the surface of the sphere and by integrating in r between r'' and r', or

$$W = \frac{Q^2}{8\pi}\int_o^{4\pi} d\omega \int_{r'}^{r''} \frac{dr}{r^2} = \frac{Q^2}{2}\left(\frac{1}{r'} - \frac{1}{r''}\right).$$

Now since the capacity of a sphere in air or vacuum is equal to its radius, this expression, which represents the work applied to the sphere, is equal to the difference between its *initial* and *final* energy. (See Eq. 126.) It

the original radius r'' were infinite, the original energy would be zero. The work required to reduce the radius to r' against the constantly increasing repulsion of the electrification for each element, becomes

$$W = \frac{Q^2}{2r'}$$

which is the work required to charge a sphere of radius r' with a quantity Q.

We have thus found that the function

$$dp\ dr = 2\ \pi\sigma^2\ dSdr = \frac{f^2}{8\pi}\ dSdr,$$

if integrated around the electrified body and then outward through the volume of infinite space, gives the energy of the electrified body. (In the last expression f is the force on a plus unit at the surface.) This is usually referred to as evidence that the field around the body is to be looked upon as the seat of the energy of the electrification. It does represent the mechanical operation of causing an electrified sphere of infinite radius to collapse to a finite size. During this operation the surface sweeps through every point in space external to the resulting spherical surface, and the equation itself does not necessarily suggest anything more, concerning space, than is involved in this simple mechanical operation.

91. *Energy of an electrical system in terms of the medium.*

The energy of an electrified sphere can also be expressed in terms of the flow of induction, and the pervance or its reciprocal the diviance of the surrounding medium through which the flow takes place. Thus we have for the energy of a charge Q on a sphere of radius r, when surrounded by a medium of perviability μ,

$$\frac{Q^2}{2\mu r} = \frac{16\,\pi^2\,Q^2}{32\,\pi^2\,\mu r} = \frac{\left[\int \mu f dS\right]^2}{8\,\pi}\,\frac{1}{4\,\pi\,\mu\,r} \quad \ldots\ldots\ldots\ldots (127).$$

The first term of this equation is the potential energy of the sphere in a medium wherein its capacity is μr. In the final term $\mu\, FdS$ represents the number of lines of induction which in air becomes FdS. The integration in this member is to extend entirely around the charge Q. If we represent the total flow of induction by I, and the diviance of the space around the body by R, then we have (118) (120)

$$I = 4\,\pi\, Q = \int \mu\, FdS$$

$$\frac{1}{P} = R = \frac{1}{4\,\pi\,\mu\, r}$$

Hence

$$\frac{Q^2}{2\,\mu\, r} = \frac{I^2\, R}{8\,\pi} \quad \text{...} (128).$$

The first member of this equation represents the potential energy of a charge in equilibrium. It appears that it is expressible in terms entirely similar to those in which we express the work per second done on a conductor of resistance R through which flows a current of electricity I. We may apply this method also to determine the energy of a system consisting of two charged spheres in air having radii r_1 and r_2 and with charges $+ Q_1$ and $- Q_2$ in which numerically $Q_1 > Q_2$. The potentials of the spheres will be

$$V_1 = \frac{Q_1}{r_1}$$

$$V_2 = - \frac{Q_2}{r}$$

Proceeding from a definite area of the charge Q_1 there are $4\,\pi\,(Q_1 - Q_2)$ lines of force which go to an infinite

distance. The diviance of the tube through which these lines flow is by Ohm's law

$$4\pi(Q_1 - Q_2) = \frac{\frac{Q_1}{r_1}}{R_1}$$

$$\text{or } R_1 = \frac{1}{4\pi r_1} \frac{Q_1}{Q_1 - Q_2} \ldots\ldots\ldots\ldots (129).$$

In Fig. 44 $P_0 M a$ is the limiting line of force which, when revolved about the axis of figure of the system, generates a surface. The tube whose resistance or divi-

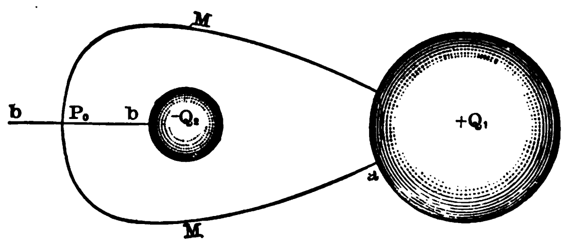


Fig. 44.

ance is determined by Eq. (129) is external to this surface. The base of this tube on the larger sphere has an area $\frac{Q_1 - Q_2}{Q_1} 4\pi r_1^2$, and the charge on the base of the tube is $Q_1 - Q_2$. Compare Fig. 26.

The charge on the larger sphere within the critical surface is $+Q_2$. From this charge $4\pi Q_2$ lines proceed to the smaller charge on the other sphere. The difference of potential between the two spheres is $\frac{Q_1}{r_1} + \frac{Q_2}{r_2}$. Hence as before

$$4\pi Q_2 = \left(\frac{Q_1}{r_1} + \frac{Q_2}{r_2}\right) \frac{1}{R_2}$$

$$\therefore R_2 = \frac{1}{4\pi r_1} \frac{Q_1}{Q_2} + \frac{1}{4\pi r_2} \ldots\ldots\ldots\ldots (130).$$

This is the diviance of the tube of force within the critical surface and terminating on the two bodies. The first term of (130) is the diviance to the lines within the critical surface proceeding from the charge $+ Q_2$ on the larger sphere, on the assumption that they proceed to an infinite distance, precisely as in Eq. (129). $\frac{Q_2}{Q_1}$ is the fraction of the surface occupied by the bound charge Q_2. In fact if the first term of (130) and the diviance R_1 (Eq. 129) be considered as resistances in multiple, the product of the two divided by their sum will be found to be $\frac{1}{4 \pi r_1}$. This would be the diviance to all the lines proceeding from sphere r_1 if its lines were radial. Eq. (101.) The remaining part of the diviance R_2 is the term $\frac{1}{4 \pi r_2}$. This is the diviance around sphere r_2 if it were alone in space and its lines of flow were radial. (Eq. 101.) It is evident that r_1 and r_2 are entirely independent of Q_1 and Q_2. If $Q_1 = Q_2$ then

$$R_1 = \infty$$

$$R_2 = \frac{1}{4 \pi r_1} + \frac{1}{4 \pi r_2} \text{ ....(131).}$$

If $r_1 = r_2 = r$

$$R_2 = \frac{1}{2 \pi r}$$

which is identical with (103). If we apply Eq. (128) to the flow through the two tubes whose diviances are given in (129) and (130) we have

$$W = \frac{1}{8 \pi} (I_1^2 R_1 + I_2^2 R_2)$$

$$= \frac{1}{8 \pi} \Bigg[16 \pi^2 (Q_1 - Q_2)^2 \frac{1}{4 \pi r_1} \frac{Q_1}{Q_1 - Q_2}$$

$$+ 16\,\pi^2\,Q_2^2\left(\frac{1}{4\,\pi r_1}\,\frac{Q_1}{Q_2}+\frac{1}{4\,\pi r_2}\right)\Bigg]$$

$$=\frac{Q_1^2}{2r_1}+\frac{Q_2^2}{2r_2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(132).$$

By reference to (126) it will be seen that this is the sum of the energy of the two spheres, in terms of their respective charges and capacities. It is, therefore, evident that the results reached in this article are a restatement in other terms of those reached in section (86) as summed up at the close of that section.

It is evident that the first term of (130) is the diviance from the larger sphere to the surface of zero potential between the two spheres. The second term is the diviance of the rest of the tube.

92. *Electrostatic determination of resistance.*

Assume a conducting sphere of radius r, charged to a potential V, and connected with the ground through a large resistance R. A current will flow through R to the ground. This would cause the potential of the sphere to fall, since $V=\frac{Q}{r}$. Suppose the radius of the sphere to simultaneously shorten due to external compression. This would cause the potential to rise. Let the velocity of change in the radius be such that the potential remains constant. In that case the terminals of R having a fixed difference of potential, a constant current would flow through the wire R. We have then by the imposed conditions

$$V=\frac{Q}{r}=\frac{Q-dQ}{r-dr}=\frac{dQ}{dr}=\frac{dQ}{dt}\,\frac{dt}{dr}$$

But $\frac{dQ}{dt}=I$, or the current flowing in the wire.

$$\therefore\ I=\frac{V}{\frac{dt}{dr}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(133).$$

By Ohm's law it is evident that

$$I = \frac{V}{R} \quad \text{.........................} (134)$$

and by (133) and (134) we have

$$R = \frac{dt}{dr} = \frac{1}{v} \quad \text{.........................} (135).$$

The resistance of the wire is, therefore, equal to the reciprocal of the velocity of contraction required to maintain the potential of the sphere constant. The conductance of the wire is, therefore, equal to this velocity. If when the radius shortens at the rate of one-half centimetre per second the potential is found to be constant, then the resistance in electrostatic units of the connecting wire is $R = 2$. As will be shown later this would be $2 \times (3 \times 10^{10})^2 = 1.8 \times 10^{21}$ electromagnetic units ($C. G. S.$) $= 1.8 \times 10^{12}$ Ohms. According to Ayrton and Perry, the resistance of a column of gutta-percha one centimetre in section and one centimetre long at a temperature of 24° C. is 92.2 electrostatic units. Therefore if R is to be 2, the circuit may be made up of any metallic wire with a column of gutta-percha in circuit, the section of the gutta percha being 1 square centimetre and the length $\frac{2}{92.2}$ cm.

93. The results of the last section may also be reached in the following manner.

Let $t' - t''$ be the duration of the operation, during which the radius of the sphere shortens at the uniform velocity v from r'' to r'; then

$$r'' - r' = v\,(t' - t'') \quad \text{.........................} (136).$$

Further

$$V = \frac{Q}{r} = \frac{4\,\pi\,r^2\,\sigma}{r} = 4\pi\,r\,\sigma$$

$$\therefore\ \sigma = \frac{V}{4\pi r}.$$

By section 90, the force acting normally outwards on an element dS of the electrified surface is

$$dp = 2\,\pi\sigma^2\,dS$$

$$\text{where } dS = r^2\,d\omega$$

$d\omega$ being the solid angle subtended by the element dS. The energy applied to the element dS when the radius is shortened dr is

$$d^2\,W = dpdr = 2\,\pi\sigma^2 dSdr$$

$$= \frac{V^2}{8\pi}\,drd\omega.$$

Therefore the total energy applied to the surface during the operation is

$$W = \frac{V^2}{8\pi}\int_{r'}^{r''} dr \int_{o}^{4\pi} d\omega$$

$$= \frac{V^2}{2}\,(r'' - r').$$

This is the energy added to the sphere during the operation. But the initial energy of the sphere was $\frac{1}{2}\ r''\ V^2$ and at the end is $\frac{1}{2}\ r'\ V^2$, so that the energy at the close *is less* than at the beginning by $\frac{V^2}{2}\,(r'' - r')$, during

which time the energy added to the sphere was also $\frac{V^2}{2}(r''-r')$. The total loss of energy was, therefore,

$$W = V^2(r''-r') \text{..................} (137).$$

The current in the wire was

$$\frac{dQ}{dt} = \frac{V}{R}$$

$$\therefore Q'' - Q' = \frac{V}{R}(t' - t'').$$

The energy developed in the wire as heat was

$$W = V(Q'' - Q') = \frac{V^2}{R}(t' - t'')$$

or by (136)

$$W = \frac{V^2}{R}\frac{r''-r'}{v} \text{..................} (138).$$

Equating (137) and (138)

$$R = \frac{1}{v}$$

In electrostatic units, therefore, conductance is a velocity. Its unit is the centimetre per second.

94. *Reciprocal Action of two electrified bodies.*

We will denote the bodies as A_1 and A_2. Let them be insulated and neutral. Now place unit charge on A_1 and call its potential p_{11}. From A_1, over and within which this potential is constant, the potential will fall off

in every direction to zero at an infinite distance. Lines of force lead from A_1 to an infinite distance, and equipotential surfaces enclosing A_1 may be drawn at right angles to the lines of force. Body A_2 will be in this region of varying potential, but its potential is necessarily constant, at every point. This condition is brought about by a negative charge

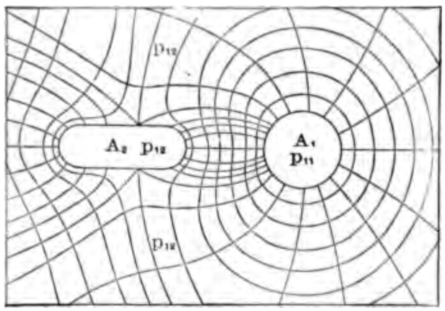


Fig. 45.

which accumulates on the side of A_2 which faces A_1 and which is so distributed as to bring the potential of that side of the body down, and an equal negative charge on the other side of A_2 which serves to bring the potential of that part up, to the resulting value. This distribution may be considered as the result of two superposed electrifications such as have been treated at length (section 59) in case of a sphere acted upon by an external point. These charges also distort the surrounding field. A certain number of lines of force originating in body A_1 terminate in the negative charge on body A_2 and continuing as lines of induction through A_2 re-appear as lines of force in its positive charge and proceed to an infinite distance. One of the surfaces of equal potential surrounding A_1 cuts A_2 in its line of zero charge and widens out into a pocket or region of constant potential coincident with body A_2. The equipotential surfaces and lines of force are shown in sectional view in Fig. 45. The equipotential lines of the figure are to be considered as intersections of the plane of the paper with surfaces of equal potential. For purposes of illustration we may consider them as lines of equal level, or contour lines around a hill, the summit of which is the high plateau A_1. A_2 is then to be considered as a level region on the *side of* the hill, and produced by cutting and filling. *The level of* this plat A_2 must then necessarily be the

same as that of some line that encircles A_1. If the distance between the bodies is large, the grading necessary to produce the level surface A_2 will be inappreciable. There will then be no appreciable distortion of the field by A_2. This was the condition assumed in the discussion relating to Fig. 44. The distortion of the field of A_1 due to the presenee of A_2 may also be considered in another way.

If A_2 were not present, the lines of force would flow from A_1 at right angles to its surface and proceed outwards to an infinite distance, distributing themselves in space according to Ohm's law. A_2 is a perfect conductor for the force, and a certain amount of the flow deflects through this body, exactly as water forced from A_1 through a porous medium would drain into a perfectly conducting cavity A_2. Such a cavity would converge upon itself the drainage lines over a definite range, and would cause a re-arrangement of all the stream lines proceeding from A_1. The water draining into A_2 from one side, would leave at the opposite side. The line of no outward or inward flow in the surface of the cavity would correspond to the line of no electrification on the body A_2. The pressure within A_2 would be constant, and the same as in a surface of equal pressure surrounding the source A_1, and intersecting body A_2 in the line of zero flow.

In determining the energy of the system in which body A_1 has unit charge upon it and body A_2 has equal charges of unlike signs induced upon it, neglecting the reacting inductions upon A_1 it is evident that the presence of A_2 does not change the flow of force from A_1. This flow is 4π. By Ohm's law.

$$4\pi = \frac{V}{R}.$$

It is evident that the charges on A_2 act differentially on A_1 the negative charge diminishing and the equal positive charge increasing its potential. Since the negative charge is nearer to A_1 than the positive, the potential V_1 is on the whole lower when A_2 is present than when it is

away. It, therefore, follows that the diviance is lower in the same ratio. The perviance of the medium around A_1 is increased by the presence of A_2. The capacity of A_1 is, therefore, also greater. It takes a greater charge to raise the potential of A_1 from zero to unity when A_2 is present than when it is away. The energy of A_1 having unit charge is, therefore, diminished by the presence of A_2. The energy of A_2 is, however, zero, since the total charge upon it is zero. The total flow of force either from it or to it is zero.

95. *Continuation.*

The potential of A_1 having unit charge being p_{11}, let the potential of A_2 due to the same charge be p_{12}. If now the charge on A_1 be increased to Q_1, the potentials of A_1 and A_2 will be $p_{11}\ Q_1$ and $p_{12}\ Q_1$. Similarly if the system be reduced to its original condition, and a charge Q_2 be placed on A_2 the potentials of A_2 and A_1 will be $p_{22}\ Q_2$ and $p_{21}\ Q_2$. If these states be simultaneously imposed, the potentials of the two bodies will be

$$\left.\begin{aligned} V_1 &= p_{11}\ Q_1 + p_{21}\ Q_2 \\ V_2 &= p_{12}\ Q_1 + p_{22}\ Q_2 \end{aligned}\right\}\text{................}(139.)$$

The coefficients in these equations are defined by the equations themselves. The coefficeint p_{11} is defined by the two conditions

$$\left.\begin{aligned} &Q_2 = o \\ &\frac{Q_1}{V_1} = \frac{1}{p_{11}} = C_{11} \end{aligned}\right\}\text{..................}(140.)$$

The condition $Q_2 = o$ does not mean that body A_2 has no electricity upon it, but that it has equal charges of unlike signs. This condition implies then that A_2 is insulated. *The reciprocal* of p_{11} is the capacity of A_1 in the presence *of A_2 insulated*, and measures the charge necessary to be

added to A_1 in order to raise its potential by unity in presence of A_2 insulated. Similarly the conditions defining p_{21} are

$$\left.\begin{aligned} Q_1 &= o \\ \frac{Q_2}{V_1} &= \frac{1}{p_{21}} = C_{21} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(141).$$

C_{21} or the reciprocal of p_{21} is numerically equal to the charge which must be put on Q_2 in order to raise the potential of the other body from zero to unity, both being insulated. p_{11} and p_{22} are quantities of the same character and are defined by reciprocal conditions like Eq. (140) while p_{12} and p_{21} are defined by conditions like (141).

Equations (139) may be solved for Q_1 and Q_2. We have by simple elimination,

$$\left.\begin{aligned} Q_1 &= \frac{p_{22}}{p_{22}p_{11} - p_{12}p_{21}} V_1 + \frac{p_{21}}{p_{12}p_{21} - p_{22}p_{11}} V_2 \\ Q_2 &= \frac{p_{12}}{p_{12}p_{21} - p_{22}p_{11}} V_1 + \frac{p_{11}}{p_{22}p_{11} - p_{12}p_{21}} V_2 \end{aligned}\right\} \quad (142)$$

which may be written

$$\left.\begin{aligned} Q_1 &= q_{11} V_1 + q_{21} V_2 \\ Q_2 &= q_{12} V_1 + q_{22} V_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(143).$$

Here q_{11} is defined by the conditions

$$\left.\begin{aligned} V_2 &= o \\ \frac{Q_1}{V_1} &= q_{11} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(144).$$

If $V_2 = o$ then body A_2 is grounded. q_{11} is the capacity of body A_1 in presence of A_2 grounded, while $\frac{1}{p_{11}} = C_{11}$

is its capacity in presence of A_2 insulated and, therefore, with zero charge.

Similary Q_{21} is defined by the conditions

$$\left.\begin{aligned} V_1 &= o \\ \frac{Q_1}{V_2} &= q_{21} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(145).$$

q_{21} is the charge on body A_1 when grounded, and A_2, insulated, is raised to unit potential in its presence. It is the capacity of A_1 for induced charge.

96. *Energy of the System A_1 and A_2.*

The energy of the system is

$$W = \frac{1}{2} Q_1 V_1 + \frac{1}{2} Q_2 V_2 \ldots\ldots\ldots\ldots\ldots\ldots(146).$$

By combining with (142) or (143) this may be expressed either in terms of the charges or of the potentials. In this way we have

$$W = \frac{1}{2} p_{11} Q_1^2 + \frac{1}{2}(p_{21} + p_{12}) Q_1 Q_2 + \frac{1}{2} p_{22} Q_2^2 \ldots(147).$$

$$W = \frac{1}{2} q_{11} V_1^2 + \frac{1}{2}(q_{21} + q_{12}) V_1 V_2 + \frac{1}{2} q_{22} V_2^2 (148).$$

If charge Q_1 be increased by dQ_1, Q_2 being constant, the work done will be

$$V_1 dQ_1 = (p_{11} Q_1 + p_{21} Q_2) dQ_1 \ldots\ldots\ldots\ldots\ldots(149).$$

The increment of energy may also be found by differentiating (147) from which

$$dW = p_{11} Q_1 dQ_1 + \frac{1}{2}(p_{21} + p_{12}) Q_2 dQ_1 \ldots\ldots\ldots(150).$$

Equating (149) and (150) and reducing we have

$$p_{21} = p_{12} \quad \text{..(151).}$$

Hence, whatever may be the shape or size of two insulated bodies, if either receive a unit charge, the potential of the other body due to that charge will be reciprocally the same.

From conditions imposed it is evident that $p_{11} > p_{12}$ and $p_{22} > p_{21}$. Therefore by (142) q_{11} and q_{22} of (143) are essentially positive and q_{21} and q_{12} are essentially negative and equal to each other. The last statement means that the negative charges induced on either body, when grounded, in the presence of the other body raised to unit potential, are reciprocally equal.

97. *Perviance of tubes of force.*

If two tubes of force originate on any charge in equilibrium on a conductor, and terminate at an infinite distance and have equal charges Q upon their bases, the tubes will have equal perviances. In Fig. 46 the tube originating at the place of greatest curvature, where the density is great, will have a small base. Since the bounding lines of the tube are normal to the surface, the tube diverges more rapidly than is the case with tubes originating at places where the curvature is less. The portions of such tubes intercepted between any two surfaces of equal potential will have equal perviances.

Fig. 46.

In Fig. 47 one tube is shown originating on a body having a positive charge and terminating at an infinite distance where the potential is zero. Another tube is *shown originat-*

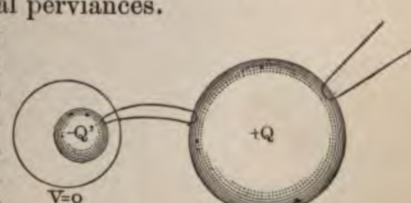


Fig. 47.

nating in the same body and having an equal charge on its base. This tube proceeds to a negatively charged body. Around this body there must be a closed surface of zero potential, within which the potential is negative. The perviance of this tube between the positively charged body and the surface of zero potential is equal to that of the infinite tube before described.

CHAPTER VI.

MAGNETISM.

98. Most of the definitions given in the introduction apply equally to electrical and to magnetic discussions. We proceed to point out the nature of the distinctions that must be made.

A mass of iron when placed in a field of magnetic force, behaves somewhat like an imperfect conductor, when placed in a field of electric force. The iron becomes polarized by the magnetic stresses; surface charges of magnetism of unlike signs accumulate on opposite sides, and for points within the iron, the field due to these surface charges opposes that of the inducing field. The resulting force within the iron is, therefore, less than it would be at the same point in space if the iron were away. As in the case of dielectrics, the force due to these surface distributions is not so great as that of the inducing field, so that the resulting force is not zero, as it is in a perfect conductor in a field of electric force. Since lines of force penetrate the mass of the iron, the magnetic potential within the iron cannot be constant. Internal cavities have a field of force, and their walls have a magnetic distribution, just as an internal cavity in a mass of shellac placed in a uniform field would have distributions of electricity of unlike signs on opposite faces where the lines of force enter and leave the cavity. The magnetic charges cannot be removed at all by conduction.

The distribution of magnetic lines of force around a mass of iron placed in a field is such, that their form and arrangement may be ascribed to a variation in the conducting power of the medium for lines of force.

Iron is by far the best conductor for the flow of magnetic force. Other media such as brass, copper, shellac, etc., behave very nearly like air.

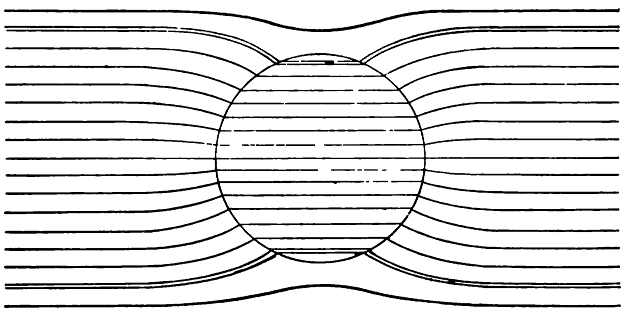


Fig. 49. Sphere having permeability 2.8 in a uniform magnetic field. (Kelvin.)

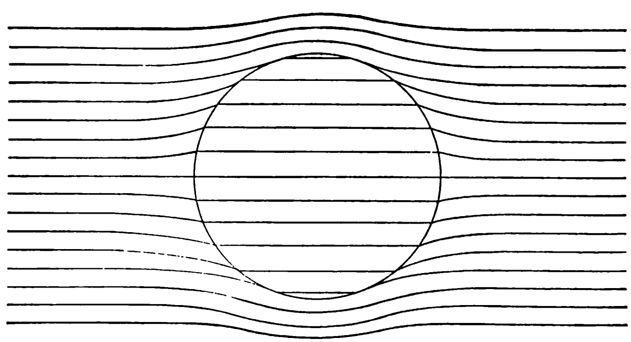


Fig. 50. Sphere having permeability 0.48 in a uniform magnetic field. (Kelvin.)

Fig. 49 shows the distorting effect of a spherical mass of matter having a permeability 2.8, placed in a uniform field. Before the introduction of the iron, the field would be represented by parallel straight lines, equally spaced. The direction of the lines at any point is determined by *combining* the forces due to the distributions of north

and south magnetism which accumulate on the two hemispheres, with the force of the original field.

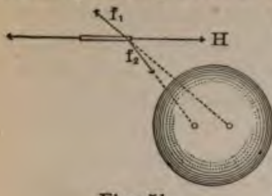


Fig. 51.

In Fig. 51 H represents the force of the original field on the north pole of a needle. f_1 and f_2 represent the forces due to the polarities of the sphere. When these three forces are combined, the resultant F will represent the direction of the resulting force.

When such constructions are made at various points around the sphere, the resulting F at any point is seen to be tangent to lines of a system like that shown in Fig. 49. The behavior of the iron mass shown in Fig. 49 may also be described in another way. These lines of magnetic force distribute themselves according to Ohm's law. The iron is a better conductor for the magnetic field than air, and the lines of flow of the force converge on the conducting sphere. If water were being forced through a porous medium, the flow taking place in straight lines, these lines would be similarly disturbed by a spherical cavity cut into it.

The specific conducting power of different media for magnetic lines of induction is called *permeability*, which is a property similar to what we have called *perviability* for electric lines of force. The permeability is usually indicated by μ. Its numerical value for iron is as great as 2500, as will be shown later. For diamagnetic bodies, the permeability is less than unity, which is the value in free space. A sphere of such material acts like an obstruction to the flow of force. The lines of force are diverged as they would be by a sphere of less porous medium as is shown in Fig. 50, which corresponds to a permeability 0.48.

In passing from medium to medium, the number of lines of induction $\mu\ F d\ S$ in a tube is constant, as is the case with electrical induction in dielectrics. As already stated in other words, there are no magnetic bodies in which $\mu = \infty$ and therefore $F = 0$, within the body, and which would *correspond* to exceedingly good conductors

like copper. A copper ball receiving a charge of electricity by momentary contact at one of its points, conducts the charge over its surface so quickly that in an exceedingly small part of a second the distribution has become so nearly uniform, and the potential has, therefore, become so nearly constant, that no difference of charge or potential can be detected by direct experimental means. Therefore under such circumstances we may treat copper as a perfect conductor of electricity and of electrical induction. This is, however, entirely consistent with the idea that will be developed later, that a permanent condition of either equilibrium or flow is never really reached in a finite time, and that under other circumstances, copper cannot be treated as a perfect conductor.

Magnetism is developed in iron and other bodies by stresses due to a field of force. The repelled polarity cannot be removed by any means while the attracted polarity is retained. The sum of the magnetic quantities on any magnetized body is always zero. The north and south magnetism is always present in equal quantity and of unlike sign.

The ultimate particles are apparently permanent magnets, which when the iron solidifies from fusion, so arrange that the polarities of adjacent particles are mutually satisfied, so that they exert equal and opposing forces at external points. In very weak fields these mutual attractions are not much changed, but with fields of increasing strength the particles finally begin to yield in a very general manner, as if a molecular panic had been set up. This is shown by the rapid rate of increase in the strength of the magnet as the strength of the field increases. When this rate of increase is at its maximum, a maximum number of the elementary magnets have their axes at right-angles to the lines of the inducing field, in which position they are most sensitive to change. This happens in soft iron when the field is about twelve or fifteen times the strength of the earth's *horizontal* component in middle latitudes. Thereafter *the effect of* increasing the field diminishes.

A line of particles having their north poles turned in the same direction is called a magnetized filament.

99. *To find the potential at any point due to a magnetized filament.*

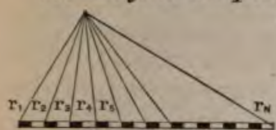


Fig. 52.

The potential at any point is the sum of the potentials due to each elementary pole m. This is

$$\frac{m}{r_1} - \frac{m}{r_2} + \frac{m}{r_2} - \frac{m}{r_3} + \frac{m}{r_3} - \frac{m}{r_4} \cdots\cdots \frac{m}{r_n} = \frac{m}{r_1} \frac{m}{r_n} \quad (152).$$

The potential is thus due to the poles at the ends of the filament. It will be positive if the nearer end terminates in a positive or north polarity and *vice versa*. At points where $r_1 = r_n$ the potential will be zero. If the ends of the filament are joined so that a closed ring is formed, then $r_1 = r_n$ everywhere, and the filament will show no polarity.

100. *Magnetic Susceptibility. Permeability.*

If a long and thin wire having a section S be placed along the lines of a uniform magnetic field of strength H, poles of strength m are formed at each end, by induction. Lines of flow of the field converge on the wire, as upon a channel of low resistance. The space occupied by the wire has a larger flow than it would have were the iron removed. The additional flow through the section at the middle of the wire is represented by $4\,\pi m$, where m is the quantity of magnetism which develops on one end of the wire. If σ represent the density of the magnetic separation across the middle section of the wire then $\sigma S = m$. If B represent the number of lines of induction carried by the middle section, H being the strength of the force over the same section, then

$$SB = SH + 4\,\pi\, m \ldots\ldots\ldots\ldots\ldots \ldots\ldots (153).$$

$$\text{or } B = H + \frac{4\pi m l}{S l}$$

$$= H + 4\pi I. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(154).$$

Where I is the magnetic moment per unit volume, of a short piece of wire at the middle having a length l. The quantity I is called the intensity of magnetization of the iron.

It is evident that B, H and I are quantities of the same kind or dimensions. Therefore we may write

$$I = K H \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(155)$$

where K must be an abstract number, or a numeric. It is defined by the equation

$$K = \frac{I}{H}.$$

It is the average increase in the intensity of magnetization per unit increase in the strength of the inducing field. It is called the *magnetic susceptibility* of the iron. In free space $K = 0$. In diamagnetic substances it is negative but small. No diamagnetic substance is known in which K is numerically as great as $\frac{1}{4\pi}$. Therefore B is always positive, and we may write

$$B = H(1 + 4\pi K) = \mu H \ldots\ldots\ldots\ldots(156).$$

In diamagnetic bodies μ is less than unity while in para-magnetic bodies it is greater than unity.

Diamagnetic bodies offer greater resistance to the flow than space (See Fig. 50), while para-magnetic bodies have greater conducting power than free space. (Fig. 49.) The quantity μ is called permeability. It varies greatly for different irons, and for the same iron it varies with B or H. It also depends on whether H is increasing,

or diminishing, although vibrations and jars tend to obliterate such differences. Fig. 135 shows Ewing's values for μ and B in terms of H, and Fig. 136 shows his determinations of K and I. The determinations were all made on the same specimen of soft iron wire by methods that will be discussed later.

101. *Lines of Force and Lines of Induction.*

A mass of iron like a long cylinder of circular section, placed with its axis parallel to the lines of force, becomes magnetized throughout. Its external surface, however, acquires properties quite different from interior sections which are bounded on both sides by iron. On the external surface are distributions of free magnetism, those on opposite ends having opposite signs. The opposite distributions are separated by a line of zero charge at or near the middle of the rod, and the density of free magnetism diminishes in a very complex manner from points at or near the ends to the line of zero density.

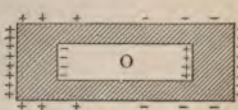


Fig. 53.

This free magnetism diminishes the strength of the inducing field within the mass as do electrical charges on a body in a similar position in an electrical field. It is the resulting field that is called H when reference is made to the force within a magnetized mass. (Eq. 153.) Suppose a cavity to be made in a magnetized mass. Let the cavity have the form of a cylinder co-axial with that of the iron mass, and parallel with the lines of the inducing field. The value of H as defined above at any point O, is the force due to the external field, and the distributions on the external surface of the iron.

The cavity having a section S we have surface charges on the ends of this cavity. The force H is increased on this account by $4\pi KH\ (1 - \cos\alpha)$ if O be taken at the middle point. Here α is the semi-angle of a cone whose base is either end of the cavity, and KH is the density of the *free* magnetism on the surfaces. In a

metal having no magnetic susceptibility, K would be zero, and these distributions would have zero density.

If a were very small, as in case of a long filamental cavity, the force due to the ends of the cavity would vanish. The force at O would then be H.

If, however, the cavity were short and of relatively large section, so that $a = 90°$, the force at O, midway between the distributions on the ends of the wafer-shaped cavity would be $4\pi KH$. The whole force at the point would, therefore, be

$$B = H + 4\pi KH$$

$$= H(1 + 4\pi K) = \mu H.$$

Here H and B are the values of the force at points infinitely near each other, but on opposite sides of the surface of the wafer-shaped cavity. H is the strength of field within the iron, or the number of lines of force to the square centimetre, and μH is called the number of lines of induction. The H lines of force in the iron yield μH or B lines of force when they discharge into the air on the opposite side of the magnetic film. μH is called the induction within the iron. $B\,d\,S = \mu\,H d\,S$ is the flow of induction in a tube of induction, of section $d\,S$. Each tube of induction is closed on itself, and however heterogeneous the material through which it threads, the induction $\mu\,H\,d\,S$ is constant throughout the tube. See Eq. (156.)

102. *To determine the strength of a magnetic field.*

A unit magnetic field is one in which unit magnet pole is urged along the lines of force with a force of one dyne. In a field of strength H, a pole of strength m will be urged with a force of mH dynes.

If a bar magnet be suspended to oscillate in a horizontal plane, it will follow the law of a compound pendulum.

Let θ be the angle of deflection of a swinging magnet

from its position of repose. Let the angular velocity at the instant be ω. Call I the moment of inertia of the magnet and its stirrup. Since the energy of the rotating mass will be $\frac{1}{2}\omega^2 I$ it follows that the increment of energy when θ diminishes by $d\,\theta$ will be $I\,\omega\,d\,\omega$. The moment of the force causing this motion is $mH\,sin\;\theta l$ where l is the length of the magnet. In imparting the increment of energy $I\,\omega\,d\,\omega$, the magnet sweeps over an angle $d\,\theta$ and the work applied is $mH\,sin\;\theta l\,d\,\theta$

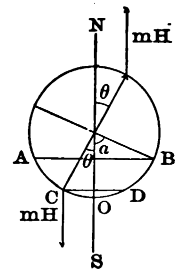


Fig. 54.

$$\therefore\; I\,\omega\,d\,\omega = -\,m\,l\,H\,sin\,\theta\,d\,\theta.$$

When $\theta = 0$ the angular velocity will be a maximum. Call it ω_0. The product $m\,l$ is called the magnetic moment of the magnet and is denoted by M.

Integrating the last equation over the semi-arc θ

$$I\int_{\omega}^{\omega_0}\omega d\omega = -MH\int_{\theta}^{0} sin\,\theta\,d\,\theta$$

or

$$\frac{1}{2}\,I\,(\omega_0^2 - \omega^2) = MH\,(1 - cos\;\theta)\ldots\ldots\ldots\ldots(157).$$

Let the greatest value of θ, when $\omega = 0$ be a. Then

$$\frac{1}{2}\,I\,\omega_0^2 = MH\,(1 - cos\;a)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(158).$$

Eliminating ω_0 in (157) and (158)

$$\frac{1}{2}\,I\,\omega^2 = MH\,(cos\;\theta - cos\;a)\ldots\ldots\ldots\ldots\ldots(159).$$

For small arcs we may put

$$\cos\theta = 1 - \frac{\theta^2}{2} \text{ and } \cos a = 1 - \frac{a^2}{2}.$$

$$\text{Since } \sqrt{1-\theta^2} = 1 - \frac{1}{2}\theta^2 \text{ (section 18)}$$

$$\therefore I\omega^2 = MH(a^2 - \theta^2) \qquad \text{..................(160).}$$

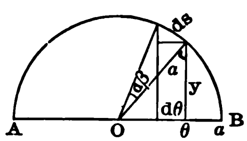


Fig. 55.

On the very short and practically straight arc $2a$ as a diameter, describe a semicircle, and from the centre lay off from O the arc θ. Then in Fig. 55 the vertical line y is

$$y = \sqrt{a^2 - \theta^2}.$$

Then by (160)

$$\omega = y\sqrt{\frac{MH}{I}}.$$

The time of describing the arc $d\theta$ is

$$dt = \frac{d\theta}{\omega} = \frac{d\theta}{y\sqrt{\frac{MH}{I}}}$$

$$= \frac{d\theta}{y}\sqrt{\frac{I}{MH}}.$$

By similar triangles in Fig. 55 calling ds the arc on the semicircle corresponding to $d\theta$

$$\frac{ds}{d\theta} = \frac{a}{y} \text{ or } \frac{d\theta}{y} = \frac{ds}{a}$$

$$\therefore dt = \frac{ds}{a}\sqrt{\frac{I}{MH}}$$

But $\frac{ds}{a}$ is a measure of the small angle marked $d\beta$. The time of describing the arc $A\ B$ is, therefore

$$t = \sqrt{\frac{I}{M\,H}} \int_o^\pi d\,\beta = \pi \sqrt{\frac{I}{MH}}$$

$$\therefore MH = \pi^2 \frac{I}{t^2} \quad \text{.........................(161).}$$

103. *Determination of the moment of inertia of the magnet.*

The moment of inertia I can be determined, by attaching a mass of non-magnetic material of known moment of inertia I'. The attached mass must have a simple form, so that I' can be computed. The time of vibration with the known moment of inertia added will be

$$t' = \pi \sqrt{\frac{I + I'}{MH}}$$

$$\therefore MH = \pi^2 \frac{I + I'}{t'^2} \quad \text{................... (162).}$$

If the experiments represented in equations (161) and (162) follow each other rapidly, so that H and M may be assumed constant then these equations give by elimination of MH,

$$I = \frac{t^2}{t'^2 - t^2} I' \quad \text{.............................(163).}$$

For determinations which require great precision, temperature corrections must be applied to M and I', and t must be corrected for the torsional effect of the suspension fibre. For the details of these corrections, reference must be made to special works.*

* See Elementary Practical Physics, Stewart and Gee.
Theory of Magnetic Measurements, Nipher.

104. *Continuation.*

In order to determine H of equation (161), another equation is necessary which may be combined with (161) by the elimination of M, and containing in addition only quantities that can be determined with precision.

For this purpose, the magnet which was oscillated in the experiment represented in (161) is replaced by another, which will be called the needle. The magnet is placed with its axis at right angles to the magnetic meridian as at m, m, so that its axis prolonged will pass through the centre of the needle. The needle is deflected and comes to rest making an angle a with the magnetic meridian. The distance r between the centres of the magnets should be five to ten times the length of the magnet.

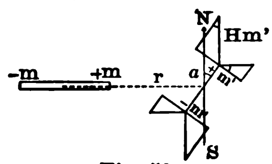


Fig. 56.

The strength of the magnet poles are m and $-m$, and of the needle m' and $-m'$ and the distance between poles of the magnet and needle are l and l'. If l' is small compared with r, the moment of the force of m on the needle is

$$A_1 = \frac{mm'\,l'}{\left(r - \frac{l}{2}\right)^2}$$

The moment of $-m$ on the needle is negative and is

$$A_2 = \frac{m\,m'\,l'}{\left(r + \frac{l}{2}\right)^2}$$

The resultant moment is

$$A = \frac{m\,m'\,l'}{\left(r - \frac{l}{2}\right)^2} - \frac{m\,m'\,l'}{\left(r + \frac{l}{2}\right)^2}$$

or

$$A = m\,m'\,l'\left(\frac{2r\,l}{\left(r^2-\frac{l^2}{4}\right)^2}\right).$$

Expanding the denominator, and neglecting a small term of second order,

$$A = \frac{2ml\,m'\,l'\,r}{r^4\left(1-\frac{1}{2}\frac{l^2}{r^2}\right)} = \frac{2\,M\,M'}{r^3\left(1-\frac{1}{2}\frac{l^2}{r^2}\right)}.$$

This represents the moment of the force with which the magnet acts on the needle, when the latter is in the magnetic meridian. When the needle is deflected over an angle a this moment becomes

$$\frac{2M\,M'}{r^3\left(1-\frac{1}{2}\frac{l^2}{r^2}\right)}\cos a \text{....................}(164).$$

When in this position, the moment of the earth's force on the needle will be

$$H\,m'\,l'\sin a = H\,M'\sin a \text{............}(165).$$

For equilibrium, the angle a must be such that these moments (164) and (165) balance, or

$$\frac{M}{H} = \frac{1}{2}\,r^3\tan a\left(1-\frac{1}{2}\frac{l^2}{r^2}\right) \text{...........}(166).$$

This formula involves the distance between the poles of the deflecting magnet. When the formula is more rigorously deduced, the quantity in the parenthesis in (166) is replaced by a series of the from $1-\frac{P}{r^2}+$ etc., where ordinarily the term having P may be dropped, since P is always found to be small and sometimes posi-

tive and sometimes negative in sign. The inaccuracies of adjustment are in fact thrown upon this quantity. We may then write for (166)

$$\frac{M}{H} = \frac{1}{2} r^3 \tan \alpha \left(1 - \frac{P}{r^2}\right) \text{............} (167).$$

The value of P may be determined by measuring α for several values of r the value $\dfrac{M}{H}$ remaining constant.

The equation may be put into the form

$$r^2 = P + \frac{2M}{H} \quad \frac{1}{r \tan \alpha}.$$

Computing for each r the value $\dfrac{1}{r \tan \alpha} = x$ and plotting r^2 and x, the intercept on the r^2 axis will give P. The last equation shows that P has the dimensions of an area, and that $\dfrac{P}{r^2}$ is an abstract number.

Combining (161) and (167) by division, we have

$$H = A \sqrt{\frac{I}{t^2 r^3}}$$

where

$$A = \sqrt{\frac{2\pi^2}{\tan \alpha \left(1 - \frac{P}{r^2}\right)}}.$$

This value of A is independent of any system of units. I is a mass into the square of a length. t is the square of a time and r^3 is the cube of a length. Therefore the magnitude of the unit in which H is measured is

$$H \text{ unit} = \sqrt{\frac{M L^2}{T^2 L^3}} = M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}.$$

We have also defined the strength of a field to be force per unit quantity. According to this definition, the magnitude of the unit would be directly as the magnitude of the force unit, and inversely as that of the quantity unit, or

$$H \text{ unit} = \frac{M\, L\, T^{-2}}{M^{\frac{1}{2}}\, L^{\frac{3}{2}}\, T^{-1}} = M^{\frac{1}{2}}\, L^{-\frac{1}{2}}\, T^{-1}$$

which is seen to be the same.

If H' be the horizontal component of the earth's field in foot-grain-second units, and H the strength in $C.\ G.$ S. units then

$$H = \left(\frac{M'}{M}\,\frac{L}{L'}\right)^{\frac{1}{2}} H'$$

or

$$\left.\begin{aligned} H &= 0.04611\ H' \\ H' &= 21.688\ H \end{aligned}\right\} \text{................} (168).$$

The value of the magnetic moment of the magnet may also be determined by (161) and (167).

We have

$$M^2 = \frac{\pi^2}{2}\ tan\ a \left(1 - \frac{P}{r^2}\right) \frac{Ir^3}{t^2} \text{.........} (169).$$

According to this equation, the magnitude of the unit of magnetic moment is

$$\sqrt{\frac{M\, L^2\, L^3}{T^2}} = M^{\frac{1}{2}}\, L^{\frac{5}{2}}\, T^{-1}.$$

We have also defined magnetic moment to be $m.\, l.$ According to this its unit would be

$$M^{\frac{1}{2}}\, L^{\frac{3}{2}}\, T^{-1} \times L = M^{\frac{1}{2}}\, L^{\frac{5}{2}}\, T^{-1}.$$

105. *Example.*

The time of vibration of a magnet was $t = 6.827$ seconds. When loaded with an inertia ring of brass, the axis of which coincided with the axis of rotation, the time was $t' = 8.562$ seconds. The moment of inertia of the ring was computed from the formula

$$I' = \frac{w}{2}\,(r'^2 + r''^2)$$

where $w = 812.9$ grains, or the mass of the ring, and where the radii of the ring were

$$r_1 = 0.1200 \text{ foot}$$
$$r_2 = 0.0953 \quad \text{''}$$

The moment of inertia of the ring was, therefore, 9.5812. The moment of inertia of the magnet computed from Eq. (163) was

$$I = \frac{(6.827)^2}{(8.562)^2 - (6.827)^2}\,9.5812 = 16.722.$$

Hence by (161)

$$MH = \pi^2\,\frac{I}{t^2} = 3.541.$$

The magnet was then replaced by the needle and the magnet was placed at a distance r of two feet from the needle as in Fig. 56. The angle a was then measured. The distance r was then changed to 1.75 ft. and the angle was again measured. The angles were

r	a
2 feet	2°14′ .4
1.75 feet	3°20′ .7

Substituting these values in Eq. (167), we have

$$\frac{M}{H} = \frac{8}{2} \; tan\ 2°14'.4 \left(1 - \frac{P}{4}\right)$$

$$\frac{M}{H} = \frac{(1.75)^3}{2} \; tan\ 3°20'.7 \left(1 - \frac{P}{(1.75)^2}\right)$$

from which by eliminating $\frac{M}{H}$

$$P = +\ 0.0083.$$

Hence

$$\frac{M}{H} = \frac{1}{2}\ 8\ tan\ 2°14'.4 \left(1 - \frac{0.0083}{4}\right)$$

$$= 0.156$$

Combining this with the value of MH we find

$$H = 4.762.$$

By means of the upper equation (168) the value of H if all our measurements had been made in $C.\ G.\ S.$ units instead of in foot-grain-second units would have been

$$H = 4.762 \times 0.04611 = 0.219.$$

When the torsion and temperature corrections are applied to these measurements, the value of H in English units is 4.767 instead of 4.762.

106. *Magnitude of units.*

In Article 100, intensity of magnetization of iron was defined to be magnetic moment per unit of volume. It was denoted by the symbol I in that section. Its unit has, therefore, the magnitude

$$\frac{M^{\frac{1}{2}} L^{\frac{5}{2}}\ T^{-1}}{L^3} = M^{\frac{1}{2}} L^{-\frac{1}{2}}\ T^{-1}$$

which is the same as for H. It follows, therefore, since $B = H + 4\pi I$ that B must be a quantity of the same kind as H and I. Since furthermore magnetic susceptibility is

$$K = \frac{I}{H}$$

and since

$$\mu = \frac{B}{H}$$

it follows that both susceptibility and permeability must be abstract numbers.

Whatever the units of length time and mass, the numerical values of μ and K remain unchanged, and those of B H and I will change in the same ratio.

107. *To find the potential at any point in the field of a magnetic shell.*

It is assumed that the shell is a thin plate whose magnetic field is such as would be produced by a current of electricity flowing around its circumference. The fundamental importance of such a distribution in its application to electrical measurements will be apparent as we proceed.

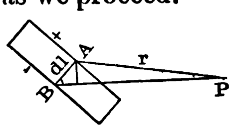


Fig. 57.

The shell may be assumed to be made up of small parallel magnets, having a length dl equal to the thickness of the shell, and having each a pole face $d\,S$ and a pole of strength $\sigma\,d\,S$. Let $A\,B = dl$ of Fig. 57 represent the axis of one of the elementary magnets. The potential at any point P due to the elementary magnet $A\,B$ is

$$dV = \sigma\,d\,S\,\left(\frac{1}{r} - \frac{1}{r_1}\right)$$

the negative magnetism being supposed on the more distant face. Calling a the angle $A\,B\,P$,

$$dV = \sigma dS\left(\frac{r_1 - r}{r_1\ r}\right) = \sigma d\ S\ \frac{dr}{r\,r_1}.$$

Since $dr = d\ l\ cos\ a$ and r and r' are sensibly equal,

$$dV = \frac{\sigma\ d\ S\ d\ l\ cos\ a}{r^2} = \frac{\sigma dl\ d\ S'}{r^2}$$

where dS' is the right section of the cone of solid angle $d\omega$, and whose oblique section is dS, and whose vertex is at P.

Hence

$$dV = \sigma dld\ \omega \qquad (170).$$

The potential at P of a shell of finite area and having a uniform distribution σ is

$$V = \sigma\ d\ l \int d\ \omega \qquad (171).$$

The quantity $\sigma\ d\ l$ characterizes the shell. It is the moment per unit area of its pole face.

At any point in the plane of the shell, or at any point whose distance r is infinite, $\int d\ \omega$ will be zero, and the potential will, therefore, be zero. At the face of the shell $\int d\ \omega = 2\ \pi$ and the potential will be $2\pi\sigma dl$. This potential will be positive at the positive face of the shell, and negative at the negative face. The difference in magnetic potential between the faces of the shell will be $4\pi\sigma dl$.

This will be the work required to carry a unit + pole from the negative face along any path around the edge of the shell to the *positive* face.

108. *To find the potential energy of a magnetic shell in a field of known strength.*

Suppose the field to be due to a pole $+m$ at the point P in Fig. 57. The work required to bring this pole to the point P from an infinite distance against the repulsion of the element magnet will be

$$dW = mdV = \frac{m\sigma\, dSdl \cos a}{r^2}.$$

This will also represent the work required to bring the shell to its position in the field of the pole m.

But $\frac{m}{r^2} \cos a\, dS$ is the flow of force from m through the tube whose oblique section is dS (article 40). Indicating by dN the number of lines of force in this tube we have

$$dW = \sigma dl dN \qquad (172).$$

Integrating over the face of the shell on the assumption that σ is uniform we have

$$W = \sigma dl N \qquad (173).$$

The potential energy of a magnetic shell in a magnetic field due to some external source is, therefore, proportional to the number of lines which intersect it. At an infinite distance from m, the energy of the shell would be zero. If the shell so moves that N increases, the energy will increase in a like ratio. The same result will follow by increasing the strength of the external field. If the shell stand edgewise in the field its energy will be zero. If the $+$ side of the shell faces $+m$ the energy will be positive. It has required energy to put the shell in that position. If the $-$ side of the shell face $+m$, the energy will be negative. In the former case looking at the shell in a $+$ direction along the lines of the field (which is the direction the north end of a needle would point) the peripheral $+$ currents around the shell which would produce its magnetization would circulate counter-clockwise.

CHAPTER VII.

ELECTRIC CURRENTS.

109. *Definition of the Unit Current.*

We have found that the potential at any point in the field of a magnetic shell is $\sigma\, d\, l\, d\, \omega$ where $d\omega$ is the solid angle subtended by the shell at the point. Suppose a current to flow around the circumference of a magnetic shell of intensity $\sigma\, d\, l$, and the current were to be so adjusted as to produce the same lines as that due to the magnetism of the shell. Let the unit in which this current is measured be so chosen that the potential due to the current is $i\, d\, \omega$. The unit current would then produce the same potential as a magnetic shell in which $\sigma\, d\, l = 1$.

110. *Potential and Strength of field on the axis of Circular Current.*

The solid angle subtended by the circular current at any point on the axis is $\omega = 2\,\pi\,(1 - \cos a)$ (Article 19). Therefore the potential is

$$V = 2\,\pi\, i\,(1 - \cos a) \ldots\ldots\ldots\ldots\ldots\ldots (174),$$

where a is the angle subtended by the radius of the circle.

The axis is a line of force due to the current. Therefore calling l the distance from the point to the plane of the circle, and r the oblique distance to the metal strip forming the conductor,

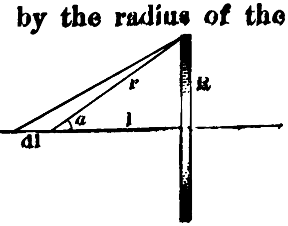


Fig. 58.

$$dV = F\, d\, l \text{ or}$$

$$F = \frac{dV}{dl} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (175).$$

(149)

The last equation shows that the strength of the field at any point on the axis may be found by differentiating (174) with respect to l. If R represent the radius of the circular current,

$$V = 2\pi i\left(1 - \frac{l}{\sqrt{R^2 + l^2}}\right)$$

$$\therefore F = \frac{dV}{dl} = -2\pi i \frac{R^2}{(l^2 + R^2)^{\frac{3}{2}}} \text{..........(176).}$$

$$= -2\pi i \frac{R^2}{r^3} \text{.......................(177).}$$

At the centre of the circle $l = 0$ and

$$F = F_0 = -\frac{2\pi i}{R} \text{....................(178).}$$

If $i = 1$ $C.$ $G.$ $S.$ unit of current or ten amperes, and F is to be 0.219, or equal to the strength of the earth's field measured in section 105, the radius of the circular current must be

$$R = \frac{2 \times 3.1416}{0.219} = 28.6 \text{ centimetres.}$$

If such a current were placed with its plane at right angles to the magnetic meridian and the current were to flow upward on the east side of the current and downward on the west side, the force at the centre due to the coil would balance that due to the earth at the point measured. The resulting force would be zero. If the current were to be reversed, the resulting force would be 2×0.219. By means of Eq. (176) the force at any distance l centimetres from the plane of the coil could be determined.

F in equations (176) or (178) represents the force in dynes, with which a unit pole would be urged along the axis at the point determined by the value of l. If $l = 0$, $R = 1$ and $i = 1$, then $F_0 = 2\pi$. Since the length of the circumference is then also 2π, it follows that each centimetre of length of a wire carrying unit current, every part of which is one centimetre distant from unit pole, will act upon the pole with a force of one dyne. This also serves to define the unit *C. G. S.* current, which is ten amperes.

111. *Action of a current having any form.*

Let df be the force with which an element of current, of length dl, and of strength i, acts on a pole of strength m, and distant r. Let β represent the angle between the line joining the pole and the element of current, and a perpendicular from m to the tangent to the circuit at the element. Then by the previous definition and from section 14,

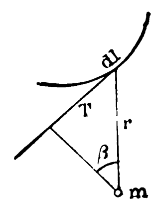


Fig. 59.

$$df = \frac{m\, i\, d\, l}{r^2} \cos \beta \quad \text{..............(179).}$$

This force urges m at right angles to the plane determined by the lines T and r. The direction of $d\,f$ reverses with the current. The force will be zero when m is on the tangent T, where $\beta = 90°$, and it will be greatest, for a given value of r, when $\beta = 0$.

The force of the entire circuit, or of any part of it, will be obtained by integrating along the wire between the proper limits. It is

$$F = m\, i \int \frac{dl}{r^2} \cos \beta \;..$$

This integration cannot be made, unless another equation, determining the form of the wire, is is given.

We may determine the force at any point in the axis of a circular current, as shown in Fig. 60. Let P be the point occupied by unit pole, whose distance from the plane of the circuit is l, and whose oblique distance from every part of the wire is r. If the current is flowing from the observer

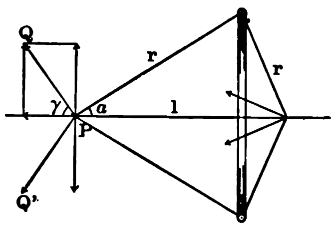


Fig. 60.

in the upper part of the wire as shown in Fig. 60, then an element $d\,l$ of the wire at that point will tend to revolve the pole P around the element in a clock-wise direction, in a circle whose radius is r. The force is represented by $P\,Q$ which is at right angles to r. If the current were reversed, this force would reverse in direction. The angle β of Eq. (180) is zero for every part of the circuit. This is known by experiment, since the magnetic lines of a straight current are concentric circles having the wire as a centre. An element $d\,l$ in the lower part of the circuit, in which the current flows towards the observer, would tend to revolve the pole in a counter-clock-wise direction. This force is represented by $P\,Q'$ in Fig. 60. The forces due to any element $d\,l$ of the wire would lie in a cone generated by revolving $P\,Q$ about the axis l keeping the angle γ constant. The sum of the components of $P\,Q$ at right angles to the axis will evidently be zero. If γ be the angle which $P\,Q$ makes with the axis, and a the angle which r makes with the same axis as in Fig. 60, the component along the axis is

$$dF = \frac{idl}{r^2} \cos \gamma = \frac{idl}{r^2} \sin a$$

$$= \frac{iR}{r^3}\, dl.$$

Hence integrating in l around the circle

$$F = 2\,\pi\,i\,\frac{R^2}{r^3}$$

which is the same as Eq. (177).

The sign of this force will depend on the direction of the current. In the case represented in Fig. 60, the force F would tend to increase the distance l, when the pole is on the left of the plane of the circuit.

112. *Magnetic Force in the field of a straight wire carrying a current i.*

The force on unit pole at m due to an element of current dl is

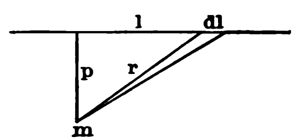


Fig. 61.

$$dF = \frac{idl}{r^2}\cos\beta.$$

Let p be the perpendicular distance from m to the wire. Then

$$\cos\beta = \frac{p}{r} \text{ and } r^2 = p^2 + l^2.$$

Substituting these values in the former equation, integrating in l between 0 and ∞ and doubling the value we have

$$F = 2\,i\,p\int_0^\infty \frac{dl}{(p^2 + l^2)^{\frac{3}{2}}} = \frac{2\,\mathrm{i}}{p} \text{..............} (181.)$$

The force varies inversely as the distance from the wire, and is constant for constant p. The lines of force are then concentric circles having the wire as a centre.

We may determine between what limits 0 and l. the

integration of (178) must be carried in order that the force may be any fraction k of $\frac{2\,i}{p}$. We have

$$2\,i\,p \frac{l}{p^2\,(p^2 + l^2)^{\frac{1}{2}}} = k\,\frac{2\,i}{p}$$

$$\therefore l = p\sqrt{\frac{k^2}{1 - k^2}}$$

$$\text{If } k = \frac{1}{2} \qquad l = \frac{p}{\sqrt{3}}$$

$$\text{If } k = \frac{99}{100},\ l = 7.01\,p.$$

Hence at a distance p of 10 centimetres from the middle point of a wire 140 centimetres in length, the force would be within one per cent of that which would be produced by a wire of infinite length. Therefore if this wire carried a current of ten amperes or one *C. G. S.* unit, the magnetic field due to 140 centimetres of the wire at a distance p of 10 centimetres from its middle point would be $\frac{99}{100}\,\frac{2}{10} = 0.198$. This is about equal to the horizontal component of the earth's field at Baltimore.

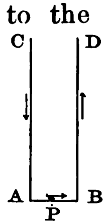


Fig. 62.

In Fig. 62 let $A\,B$ represent the wire whose effect at P has just been found. If $C\,A$ represent a vertical leading wire having a length of 500 centimetres its force at P may be computed by integrating (181) in l between 10 and 510, its distance p being 70. The integral is *not* to be doubled as has been done in (181).

We thus have

$$F_1 = 70\left[\frac{510}{(70)^2\,(70^2+510^2)^{\frac{1}{2}}} - \frac{10}{70^2\,(70^2+10^2)^{\frac{1}{2}}}\right]$$

$$= \frac{1}{70}\left(\frac{510}{514.7} - \frac{10}{70.7}\right) = 0.012.$$

This force would be directed towards the reader, as would that of the equal return wire $B\ D$, and would be opposite in direction from that of $A\ B$. The resultant field at P due to these three portions of the current would be $0.198 - 0.024 = 0.174$.

If P were at a point ten centimetres above the centre of $A\ B$ the force due to $A\ B$ would be reversed in direction. The effect due to $A\ C$ or $B\ D$ would then be found by integrating in (178) between $l = -10$ and $l = +490$.

113. *Magnetic potential in the field of a straight wire carrying a current i.*

We may apply the general equation (sections 109, 110)

$$V = \omega\, i \qquad (182)$$

where w is the solid angle subtended by the circuit.

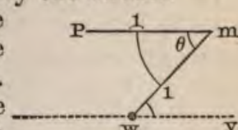


Fig. 63.

Let ω represent the section of the conductor which extends to an infinite distance above and below the paper. Practically this may mean that the wire must be eight or ten metres in length. Suppose the circuit to be completed to the left of w in a return wire, the plane of the circuit containing the line $w\ y$. The potential at any point m is determined by the solid angle ω of the cone whose vortex is at m and whose base is the circuit. The right lines drawn from m to every point of the circuit all lie in two planes which have for their traces the lines $w\,m$ and $m\,P$. The angle between these planes is

108. *To fi*
in a field of l

Suppose th
P in Fig. 5
the point P
of the elem

This will
shell to its

But $\frac{m}{r^2}$ co

tube who
ing by $d\Lambda$
have

Integr
tion tha

The
field d
to the
dista
If th
incre
incre
shel
If t
pos
po
en
tl
(

the work required to carry unit pole between two planes making with each other an angle $d\,\theta$ is

$$d\,V = 2\,i\,d\,\theta.$$

If the path is at a distance r from the wire, and has a length dS then $dS = r\ d\,\theta$. If the force be denoted by F then

$$d\,V = F\,d\,S = F\,r\,d\,\theta.$$

Equating these values of $d\ V$

$$F = \frac{2\,i}{r}$$

which is identical with (181).

114. *To find the strength of the magnetic field in the axis of a circular current of S turns.*

Let R be the radius of a conducting cylindrical shell of length l. We will suppose a current to be flowing in

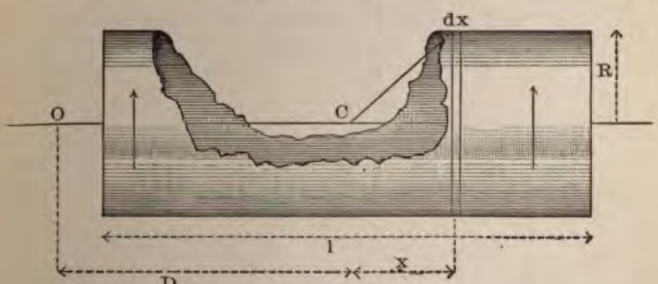


Fig. 64.

a sheet around the tube. We are required to find the magnetic force on a unit pole at a point O, whose distance from the middle point C of the tube is D. Let dx be the width of *any* element of the cylinder, whose plane

is distant x from the centre C. Let i_o be the number of units of current per centimetre, measured parallel to the axis of the cylinder, or $i_o = \frac{di}{dx}$. The force due to the current di of the element dx, at O is by Eq. (176)

$$dF = 2\,\pi\, i_o\, dx \frac{R^2}{\left((D+x)^2 + R^2\right)^{\frac{3}{2}}}$$

The force due to the entire sheet of current around the tube is

$$F = 2\,\pi\, i_o\, R^2 \int_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{dx}{\left((D+x)^2 + R^2\right)^{\frac{3}{2}}}$$

$$= 2\,\pi\, i_o\, R^2 \left[\frac{D+x}{R^2\sqrt{(D+x)^2+R^2}}\right]_{-\frac{l}{2}}^{+\frac{l}{2}}$$

$$= 2\,\pi\, i_o \left[\frac{D+\frac{l}{2}}{\sqrt{\left(D+\frac{l}{2}\right)^2 + R^2}} - \frac{D-\frac{l}{2}}{\sqrt{\left(D-\frac{l}{2}\right)^2 + R^2}}\right] \ldots\ldots(184).$$

This equation applies to any point on the axis. If $D > \frac{l}{2}$ the point O is outside of the helix. If $D = \pm \frac{l}{2}$ the point is at one end or the other of the helix and

$$F_e = 2\,\pi\,i_o \frac{l}{\sqrt{l^2 + R^2}} \quad \text{.............................(185)},$$

where $i_o\,l = i$ in case of a single current sheet around the tube and $i_o\,l = S\,i$, if S windings of insulated wire, each having a current i, cover a length l of the tube.

If $D = 0$, the point O is at the centre of the tube or helix, and

$$F_o = 4\,\pi\,i_o \frac{l}{\sqrt{l^2 + 4R^2}} \quad \text{....... ...... (186).}$$

If l is very small compared with R, as in the case of a single wire serving as a galvanometer coil, then by (185) and (186)

$$F_e = F_o = \frac{2\,\pi i_o\,l}{R} = \frac{2\,\pi\,i}{R}$$

which is the same as (178). On the other hand if l is very large compared with 4 R.

$$F_e = 2\,\pi i_o = \frac{2\,\pi\,S\,i}{l} \quad \text{.................. (187).}$$

$$F_o = 4\,\pi\,i_o = \frac{4\,\pi\,S\,i}{l} \quad \text{...............(188).}$$

The force at the ends is then half the force at the centre. (See Fig. 134.) This is due to the fact that at the centre of a long helix the two halves co-operate, but the end windings are too far away to have appreciable effect. When the testing pole is at the end of the helix, therefore, the condition is the same as it would be at the centre, were half of the helix to be removed.

For a point at the centre where $K=0$, $f(a)$ of (191) is unity for all values of a between zero and π. The area Fig. 67 represented by $\int_0^\pi f(a)\,d\,a$ is, therefore, a rectangle whose sides are unity and π and whose area is $\pi y_z = \pi$. Therefore

$$F = \frac{2\pi i}{R}$$

which is the same as (178).

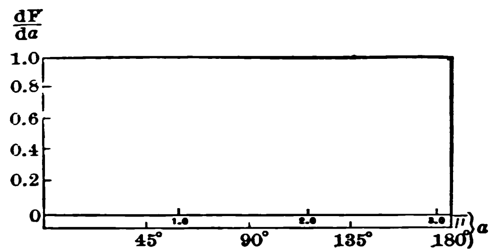


Fig. 67.

If $K=0.1$, the value of $f(a)$ of (191) may be computed for each ten degrees between 0° and 180°. The form of curve obtained for this case is shown in Fig. 68. When $a=0$ the value of

$$f(a) \text{ is } \frac{1-K}{(1-2K+K^2)^{\frac{3}{2}}} = \frac{1}{(1-K)^2}$$

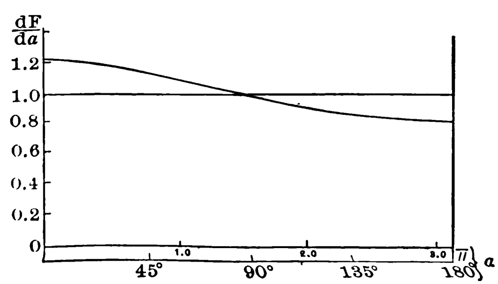


Fig. 68.

This is a measure of $\frac{dF}{da}$ or the force per unit angle at the nearest point of the wire. When $a = \pi$, $f(a)$ is $\frac{1 + K}{(1 + 2K + K^2)^{\frac{3}{2}}} = \frac{1}{(1 + K)^2}$. This quantity multiplied by $\frac{2i}{R}$ is the force per unit angle at the most remote part of the circular current. (See Fig. 68.) The intermediate values of $f\ (a)$ must be computed at a sufficient number of points so that the curve can be drawn with sufficient accuracy. For instance if $a = 20°$, the abscissa in Fig. 68 will be $\frac{20}{180}\ \pi = 0.349$ and the ordinate to the curve will be

$$f'\ (a) = \frac{1 - 0.1\ cos\ 20°}{(1 + 0.01 - 0.2\ cos\ 20)^{\frac{3}{2}}} = 1.215$$

Having plotted the curve for $K = 0.1$ from a sufficient number of such computations, the area of the curve between the limits, $a = o$ and $a = \pi$, is to be determined by means of a planimeter. In a certain case this area was found to be 2.059 square decimetres. Then the planimeter area of a rectangular figure on the same diagram was determined. The area of the rectangle of the ordinate 1.00 and abscissa π was found to be 2.046 square decimetres. Since the integral value of this rectangular area is π, we have for the integral represented by the area of the curve between o and π,

$$\pi\ y_0 = \frac{2.059}{2.046}\ 3.1416.$$

$$\text{Hence } y_0 = \frac{2.059}{2.046} = 1.006$$

$$\text{and } \int_o^\pi f\ (a)\ d\ a = 1.006\ \pi$$

Therefore at a point one-tenth of a radius from the centre, the force is

$$F = \frac{2\pi i}{R} 1.006$$

In this manner the values of y_0 for each value of K in the table below have been computed. These values are the factors by which the force at the centre must be multiplied, in order to give the force at a distance $K\,R$ from the centre.*

K	y_0	y_0 from (195)
0.0	1.000	1.000
0.1	1.006	1.008
0.2	1.032	1.033
0.3	1.072	1.076
0.4	1.145	1.145
0.5	1.246	1.250
0.6	1.425	1.413
0.7	1.687	1.687
0.8	2.234	2.231
0.9	3.92	3.841
0.95	7.17	7.035
0.99	32.07	32.512

The values of this factor y_0 as given in column two agree very well with the following empirical formula

* In case of the curve for $K = 0.99$ the value of the ordinate when $\alpha = o$ is 10000 while when $\alpha = 180°$ it is 0.252. It, therefore, becomes necessary to construct different parts of the curve to different scales.

from which the values in column three have been computed.

$$y_0 = 1 + \frac{1}{2} K \tan (K\,90)^\circ \text{ ..... } (195).$$

Assuming this formula to be sufficiently accurate we have for the force

$$F = \frac{2\,\pi\,i}{R}\left(1 + \frac{1}{2} K \tan (K\,90)\right) \text{........} (196).$$

The number of lines of force on an annulus of radius $K\,R$, radial width RdK and area

$$dS = 2\,\pi\,R^2\,KdK$$

is

$$FdS = 4\,\pi^2\,i\,RK\left(1 + \frac{1}{2} K \tan (K\,90)\right)dK$$

If we integrate this expression in K between $K = 0$ and $K = K'$ where $K'R$ is the distance from the centre, then

$$\int FdS = 2\,\pi^2\,iRK'^2 + 2\,\pi^2\,iR \int_0^{K'} K^2 \tan (K\,90)\,d\,K$$

The value of the final integral can be readily found by the planimeter method just described. Dividing out of the resulting integrals the value of K'^2 the equation may be written,

$$\int FdS = 2\,\pi^2\,i\,R\,K'^2\,(1 + [illegible] \text{......} [illegible])$$

where the values of $1 + y$ which correspond to various values of K' are given in the table below.

K'	$1+y$	K'	$1+y$
0.1	1.004	0.6	1.178
0.2	1.016	0.7	1.270
0.3	1.037	0.8	1.424
0.4	1.066	0.9	1.728
0.5	1.122	0.95	2.074
		0.99	2.985

In (197) it will be observed that $2\,\pi^2\,i\,RK'^2 = \frac{2\,\pi\,i}{R}\,\pi\,(K'R)^2$ represents the number of lines enclosed by a circle of radius $R'K$, if the force were everywhere equal to the force at the centre. If the radius of the wire be r_0 and K' be taken as $\frac{R - r_0}{R}$ then by selecting the value of $1 + y$ from the preceding table, (197) will give the total number of lines of force linking with the wire. If for example the wire radius is 0.01 of the circuit radius, then $K' = 0.99$, $1 + y = 2.985$ and the total lines linking with the wire will be

$$N = 5.85\,\pi^2\,i\,R.$$

If $i = 10$ amperes or 1 $C.\ G.\ S.$ unit and $R = 30$ cm. $N = 1732$. The difference in magnetic potential from any point on the plane along the lines of force linking with the wire, to the same point again, is $4\,\pi\,i$. (See section 113.) Calling ρ the magnetic resistance or reluctance of the magnetic circuit to these N lines we have by Ohm's law

$$5.85\,\pi^2\,i\,R = \frac{4\,\pi\,i}{\rho}$$

$$\text{or } \rho = \frac{1}{1.46\,\pi\,R}.$$

If the circular circuit have a radius $R = 30$ centimetres, the radius of the wire being 0.30 centimetre ($K' = 0.99$), i being 15 amperes, or 1.5 *C. G. S.* unit then the total number of lines N will be 2598 and $\rho = 0.0072$.

116. *Magnetic force inside a conductor.*

The force within a straight thin tubular conductor is zero. Let Fig. 69 represent such a conductor in cross section.

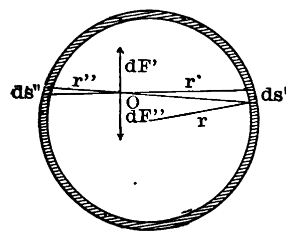


Fig. 69.

Let the current be supposed to flow in the shell, at right angles to the paper and away from the reader. If the total current be i, and the radius r then the current in each centimetre of arc will be

$$i'_0 = \frac{i}{2\pi r}.$$

Let r' and r'' be lines intersecting in a small angle at any point O, and cutting the shell at the extremities of the small arcs ds' and ds''.

The currents in the elements ds' and ds'' of the conductor are

$$di' = i'_0\, ds'$$

$$di'' = i'_0 ds''.$$

The force due to these currents at the point O will be (181)

$$dF' = \frac{2di'}{r'} = \frac{2i'_0 ds'}{r'}$$

$$dF'' = \frac{2di''}{r''} = \frac{2i'_0 ds''}{r''}.$$

By geometry we know that $\frac{ds'}{r'} = \frac{ds''}{r''}$ and by experi-

which correspond ... in the table below.

	K	
	[illegible]	1.27[illegible]
	[illegible]	1.27[illegible]
	[illegible]	1.424
	[illegible]	1.728
	[illegible]	2.074
	[illegible]	2.985

... that $2\pi^2 i RK$ ... the number of lines ... the force were every- ... If the radius of ... then by selecting ... table, (197) will ... linking with the ... of the ... = 2.985 and the

... and ... = 30 cm. ... from ... linking ... See ... we have

each centimetre of length. At the centre of the wire the force is zero. Within the wire the force is directly proportional to the distance from the centre (198).

117. *To determine the constant for a tangent galvanometer of single turn.*

The short magnetic needle is placed at the centre of the circular current where the field due to the coil may be taken as

$$F = \frac{2\pi i}{R}.$$

Therefore the force acting on each pole, m, of the needle and at right angles to the coil is

$$Fm = \frac{2\pi im}{R}.$$

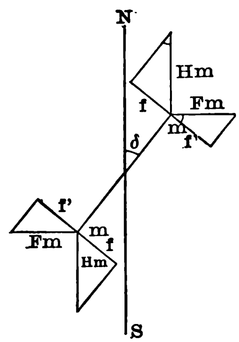


Fig. 70.

The earth's force acting on each pole of the needle and in a direction parallel to SN (Fig. 70) is Hm. Under the action of these two forces, the needle is deflected from the plane SN of the coil, the angle of deflection and equilibrium being δ. The component of Fm at right angles to the needle, and which tends to increase the angle δ is $\frac{2\pi i m}{R} \cos \delta$. The opposing component of Hm is $Hm \sin \delta$. These forces are represented in Fig. 70 by f' and f. For equilibrium, those forces are equal. Hence

$$i = \frac{HR}{2\pi} \tan \delta \text{ ....} (199).$$

A galvanometer of single turn which is to deflect its needle 45° *when carrying* a current of 12 amperes or

1.2 *C. G. S.*, the strength of the earth's field being 0.19, must have a radius R of

$$R = \frac{2 \times 1.2\pi}{0.19} = 39.68 \text{ centimetres.}$$

The field due to a circular current of one turn is shown in Fig. 71.

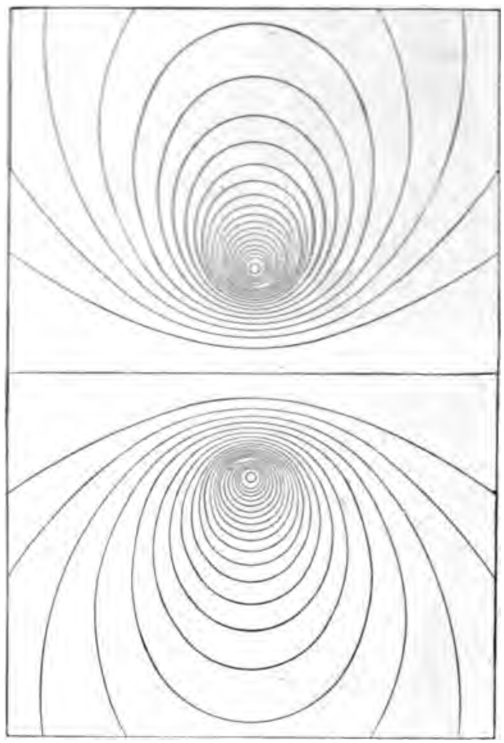


Fig. 71.

118. *Helmholtz-Gaugain tangent galvanometer.*

The needle is placed between two parallel coils, and it *is desired to* make the field as nearly constant as possible

at the point on the common axis midway between the two coils where the needle is placed (Fig. 72.) If the

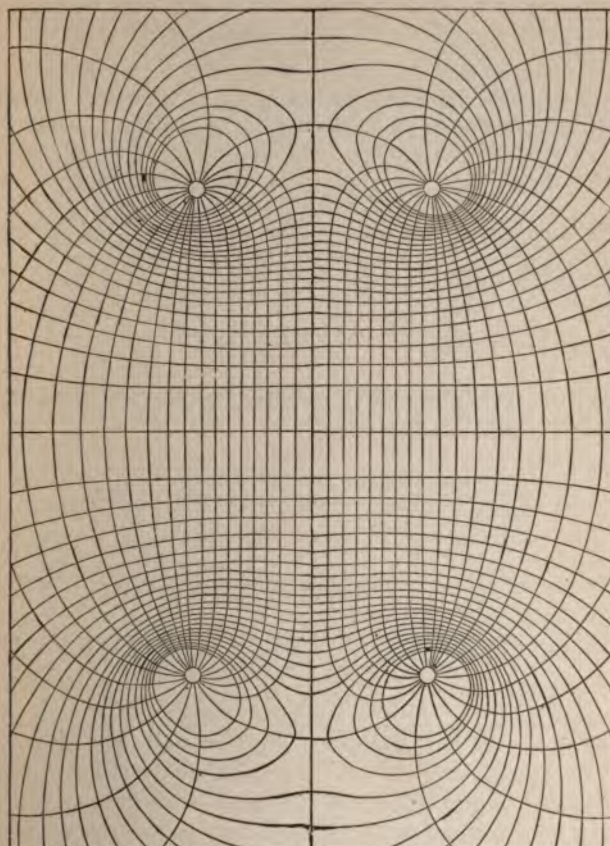


Fig. 72.

coils are too near together, the lines of force at this point will be convex towards the axis, and will diverge as they depart from the middle plane, as they do in a galvanometer of single turn (Fig. 71). If the coils are too far apart, the lines at the middle plane will be con-

cave towards the axis. The field due to both coils is (176)

$$F = 4\pi i \frac{R^2}{(l^2 + R^2)^{\frac{3}{2}}}$$

where $2l$ is the distance between the two coils. We are to adjust l so that the field will be as nearly constant as possible, or so that $\frac{dF}{dl}$ shall be a minimum.

Differentiating the last equation twice and placing $\frac{d^2 F}{dl^2} = 0$ we find

$$2l = R.$$

When the instrument is thus constructed the value of F in the last equation becomes

$$F = \frac{2\pi i}{R} \times \frac{16}{5\sqrt{5}} = 1.431 \times \frac{2\pi i}{R}.$$

The force at the centre of either coil due to the action of both will then be

$$F = 1.354 \frac{2\pi i}{R}.$$

The lines of force due to the two coils are shown in Fig. 71. They link either with one or with both wires. The lines of equal potential on this diametral plane are at R ∟ to the lines of force and are also shown in this Fig.

119. *Mutual action of coils of one winding.*

We have found (section 108) that when a magnetic shell is cut by N lines of force, the potential energy of the shell is

$$w = \sigma\, d\, l\, N. \text{ See (173).}$$

For the equivalent plane current

$$w = i\,N.$$

See Eq. (173) and section 109.

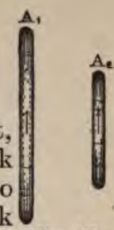


Fig. 73.

If two coils A_1 and A_2 each carry unit current, a certain number M_1 of lines due to A_1 link with A_2. If the current in A_1 be increased to i_1 the number of lines due to A_1 which link with A_2 will be

$$N_1 = M_1 i_1.$$

If coil A_2 have a current i_2 its energy would, therefore, be

$$w_2 = N_1\,i_2 = M_1\,i_1\,i_2 \quad\text{.......................(200).}$$

This would be the work required to carry coil A_2 out of the field due to A_1, in doing which the moving coil would cut $N_1 = M_1\,i_1$ lines of force.

Similarly let M_2 represent the lines due to A_2 having unit current which link with A_1. Then the number of such lines when the current in A_2 is i_2 becomes

$$N_2 = M_2 i_2.$$

The energy of coil A_1 carrying a current i_1 with which N_2 lines of force link would, therefore, be

$$w_1 = N_2 i_2 = M_2 i_2 i_1 \quad\text{...........................(201).}$$

This represents the negative work required to carry coil A_1 into the field of A_2 again, restoring the original relative positions. If we, therefore, consider the two quantities w_2 and w_1 of (200) and (201) to have unlike signs, their algebraic sum is zero, from which

$$M_2 = M_1.$$

It is, therefore, clear that the number of lines of force due to unit current in either coil which link with the other coil are reciprocally equal.

If the coils have a common current, i, then the energy of either coil due to the presence of the other is

$$w = Mi^2 \dots\dots\dots\dots (202).$$

M is called the coefficient of mutual induction of the two coils. It is equal to the number of lines due to either which link with the other, when the current is unity in the primary coil which produces the field. It is evidently also equal to the energy of either coil in the presence of the other if each carry unit current (202).

The two coils shown in Fig. 72 each have a field like that in Fig. 71. Since the currents are parallel, Fig. 72 represents the lines due to both coils. These coils attract each other and $w = M i^2$ measures the work that would be required to separate them to an infinite distance from each other. If the currents ran in opposite directions, the coils would repel each other, and the work required to cause them to approach each other would be positive in sign.

120. *Self-induction.*

The lines of force due to any coil of one winding link with the current to which they are due. Although the coil does not tend to move with respect to its own field, it is clear that such a system represents energy. If L represents the number of lines due to a coil of single turn having unit current, the energy required to produce the system composed of a current i having $L\,i$ lines of force linking with it is

$$w = \int_0^i L\,i\,di = \frac{1}{2} L\,i^2 \dots\dots\dots\dots (203).$$

If the current be doubled, the number of lines due to the current is doubled, and each line links with twice as many units of current.

This energy is applied in producing a stressed condition in the region around the wire. We shall investigate *this subject* more at length further on, but it may be well *to point* out here some of the physical peculiarities in-

volved in the formation of such a magnetic field. When a given battery or dynamo is closed through a circuit having a self-induction coefficient L, the operation of bringing the current up to a given strength i, and of producing $L\,i$ lines linking with the circuit requires a certain interval of time. The greater L is, the longer is this time. If the battery or source of electricity is cut out of the circuit by a short circuit the energy returns on the wire and manifests itself by a great increase of current. A great magnet thus charged and shut out of the circuit of the battery, without opening its circuit, can be destroyed in a few seconds by the current developed by the return to its wire of the energy of its surrounding field. The same results are seen when the field-current of a loaded dynamo is opened.

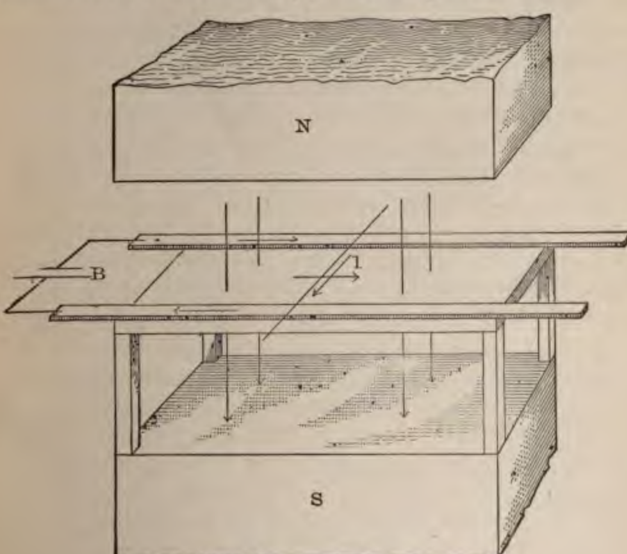


Fig. 74.

121. *Motion in a magnetic field due to an external source, of a conductor carrying a current.*

Let B in Fig. 74 represent a source of electricity,

where dN is the change in the number of lines linking with the circuit due to unit change in the current. Inserting this value of L in (215)

$$\frac{dN}{dt} = Ri \text{ .... } (217).$$

The electromotive force of discharge is, therefore, the change per second in the number of lines of force which link with the circuit. It appears from (216) that idN may replace $Lidi$ in the equation for energy (213).

124. *E. M. F. of a conductor moving in a magnetic field.*

The result reached in the last section enables us to find the *E. M. F.* of a moving conductor. If the battery B of Fig. 74 be replaced by a wire, and the slider l moved in the opposite direction from that shown in the figure, it will generate the current represented in the figure.

Let v represent the velocity of the slider, in centimetres per second, l the length of the slider in centimetres, and H the strength of the field due to N. S. Then the area swept over by the slider in one second will be lv and the number of lines of force cut per second by the wire will be

$$E = Hlv \text{ .... } (218).$$

This is the *E. M. F.* in *C.G. S.* units.

If the length of the slider be 100 centimetres and the velocity 1500 centimetres per second the area swept over per second will be 1.5×10^5. The vertical component of the earth's field is about 0.3. In such a field the *E. M. F.* generated will be $1.5 \times 10^5 \times 0.3 = 4.5 \times 10^4$, *C. G. S.* units. As we shall see later, one volt is 10^8 *C. G. S.* units. This *E. M. F.* is, therefore, 4.5×10^{-4} or 0.00045 volt.

As soon as the wire l begins to move in a direction *opposite to* that indicated in Fig. 74, and the current is

started in the direction indicated by the arrows, the interaction between current and field resists that motion, and urges l in a direction opposite to that which we force it to take. The wire tries to act as a motor, when we force it to act as a generator. We have already seen in section 121 that when we allow the wire l freedom of motion and drive it electrically, as a motor, it at once begins to act as a generator, producing an opposing current, and thus tries to stop itself.

In Fig. 75 let $A'A$ and $C'C$ represent a plan view of the rails shown in Fig. 74. They are connected only by two sliders G and M. Suppose the positive direction of the field to be away from the reader as in Fig. 74. If G be moved to the right as a generator the current thereby started will be in the direction of the arrows. The wire G resists this motion, and tries to move in the opposite direction. The current runs through the slider M. This wire will then be driven as a motor in the direction shown by the arrow. If there were no frictional resistance at the points of contact, and no other load, the wire M would move with the same speed as G. The area between the two wires and rails would remain unchanged. In that case M is cutting across the lines of the field exactly as G is doing. It is being directly moved by a different agent, but it is being moved so that it will generate an equal and opposing current. Therefore there will be no resulting current. Therefore the power applied, Ei, is zero. But no power is required to drive a frictionless machine.

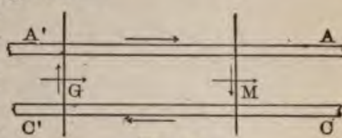


Fig. 75.

If the motor wire be restrained by a weight, $P' = m'g$ dynes, hung on a frictionless pulley, then M will move more slowly than G. The back $E. M. F.$ of the motor will be Hlv' and the resulting $E. M. F.$ in the circuit will be

$$E - E' = Hl(v - v') = iR \qquad (219).$$

If $P = mg$ represent the force applied to the wire G, the power applied to G is

$$Pv = H\,l\,v\,i = Ei \quad\text{..................(220)}$$

$$\therefore\ P = H\,l\,i \quad\text{.......................(221).}$$

The force required to drive the generator is, therefore, proportional to the strength of the field, the length of the active wire and the current it carries.

The strength of field H, in the interpolar space of the average motor or dynamo may be taken as 5,000 *C. G. S.* units. A wire in this space, 30 centimetres, or about one foot in length, and carrying a current of 100 amperes or 10 *C. G. S.* units, would be urged tangentially with a force of

$$P = 5{,}000 \times 30 \times 10 = 1.5 \times 10^6 \text{ dynes}$$

$$= 1.529 \text{ kilogramme.}$$

This is equivalent to about 3.4 lbs. for every foot of wire in the polar spaces. By (218) if the velocity of this wire be 1500 centimetres per second (or about 3,000 ft. per minute) the *E. M. F.* generated by the same wire would be

$$\frac{5{,}000 \times 30 \times 1500}{10^8} = 2.25 \text{ volts.}$$

125. *Potential energy of a current in a magnetic field due to an external source.*

This subject has been treated in sections 108 and 109. We found that the work required to bring a magnetic shell from an infinite distance where no external lines intersect it, up to a point in the field where N lines intersect it is $\sigma\, d\, l\, N$ and for the equivalent current the work is $i\,N$.

in any armature wire the power would be by (206) or (211)

$$W = E \times 2i = E\, i_a$$

$$= 2\, n\, c\, N\, i = n\, c\, N\, i_a$$

Where $i_a = 2i =$ the current delivered by the armature. This equation is identical with (230).

127. *Graphical representation of the operation of a generator and motor.*

Draw the line G, R', M to represent the resistance of a circuit containing a dynamo and a motor. The lengths of the lines G, R' and M are respectively proportional to the resistances of generator, line and motor.

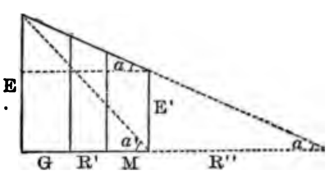


Fig. 79.

Draw the lines E and E' at right angles to the former line, and of such length that they similarly represent the electromotive forces of the generator and motor respectively. Draw a straight line through the tops of the lines E and E'. The slope of this line is determined by the equation

$$\tan a = \frac{E - E'}{G + R' + M} = \frac{E - E'}{R} = i \text{.........} (232)$$

in which R represents the entire resistance. Equation (219) shows that $\tan a$ must represent the current flowing in the circuit. If the armature of the motor is loaded so heavily that it cannot rotate, then $E' = 0$ and $E = Ri$. The current will then be represented by

$$\tan a' = \frac{E}{R} = i_o \text{.............................} (233).$$

If during this operati... ...than current is prevent...

ing of resistance, then when the motor has been brought to rest the resistance of the circuit will have been increased by some definite resistance R'' such that

$$tan\ a = \frac{E}{R + R''} = i \ldots\ldots\ldots\ldots\ldots\ldots\ldots (234).$$

Equations (232) and (234) may be written

$$Ei - Ri^2 = E'i$$

$$Ei - Ri^2 = R''i^2.$$

In these equations, Ei represents the power developed in the generator and impressed upon the circuit. Ri^2 represents the power equivalent to the heat generated each second in the wire of the machines and line, the resistance of which is R. In the latter equation $R''i^2$ represents similarly the heat which is generated each second in the resistance R''. The first of these equations represents a condition in which R'' is absent and the motor is doing work. The second equation represents a condition in which the motor is doing no work, but the heat equivalent of $R''i^2$ replaces the work formerly done by the motor. It is evident that $E'i = R''i^2$, and that $E'i$ measures the work delivered by the motor each second.

128. *Power delivered by a motor.*

Calling $E'i = w$ we have from (232)

$$w = Ei - Ri^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (235)$$

in which if R represent the resistance of the motor only, E will represent the potential difference on its terminals. Ei will then be the power delivered to the motor, Ri^2 will represent the power lost as heat generated in and radiated from the wires of the motor, and their difference w will be the power delivered by the motor.

If in this equation we place $w = 0$ and solve for i we have this condition satisfied when

$$i = 0 \text{ or } i = \frac{E}{R}.$$

The condition $i = \frac{E}{R}$ is realized when the motor-pulley is loaded so heavily that the armature cannot rotate. The condition $i = 0$ is realized when all load, even journal friction, is removed, and the back *E. M. F.*, E' of the motor increases to E.

For a given motor R will be constant, and we may regard w as a function of i and E. Eq. (235) is then the equation of a surface.

For any fixed value of E, w will be zero when the motor connot revolve, and when all load is removed and the armature revolves so rapidly that no current flows through the motor. For intermediate loads, there must be a condition of maximum output. The condition for this is obtained by differentiating (235) and is

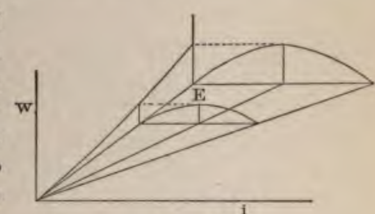


Fig. 81.

$$\left(\frac{dw}{di}\right)_E = E - 2\,Ri = 0$$

$$\text{or } i = \frac{1}{2}\,\frac{E}{R} \quad\text{..........................(236).}$$

This shows that for maximum output, the load must be so adjusted that the current is one-half that which flows through it when the motor is at rest.

When E is constant (235) is the equation of a parabola, the vertex of which is the maximum just determined.

Solving (235) for i we have

$$i = \frac{1}{2}\,\frac{E}{R} \pm \sqrt{\frac{1}{4}\,\frac{E^2}{R^2} - \frac{w}{R}}.$$

Introducing into this equation the condition of maximum of (236) by the elimination of i, we have

$$w = \frac{1}{4} \frac{E^2}{R} \quad \text{.................................} (237).$$

If we combine the same equations by eliminating E we have

$$w = R\, i^2 \quad \text{..................................} (238).$$

Equation (236) is the equation of the projection of the maximum points on the surface, on the plane of $E\, i$, while (237) and (238) are the equations of the projections of the locus of maximum on the other two coordinate planes. Each of these three equations represents the output of a motor when the condition of maximum output is satisfied.

Equation (236) asserts that the current corresponding to maximum, is one-half the current which would flow through the motor if it were stopped and the supply potential were the same. The value of this current is proportional to the value of E assumed. The projection is a right line.

Equation (237) shows that the maximum output of the motor for any value of E is one-fourth of the energy that would be delivered to the motor if it were stopped and the supply potential remained unchanged. The output itself is proportional to the square of the value of E assumed. The projection is a parabola.

Equation (238) shows that the maximum output of the motor is equal to the power lost as heat in the motor. It is, therefore, equal to half the power then delivered to the motor from the mains. The efficiency is, therefore, only fifty per cent. This output is proportional to the square of the current then flowing. The projection is a parabola. It should be observed that an efficiency of fifty per cent is not the maximum efficiency of which a motor is capable, but it is the efficiency when it is working under conditions of maximum output.

The general conditions of maximum w if it be regarded as a function of i and E are

$$\left(\frac{dw}{di}\right)_E = E - 2Ri = 0$$

$$\left(\frac{dw}{dE}\right)_i = i = 0$$

these conditions being made simultaneous. The first equation is a condition of maximum in any section of the surface determined by the condition $E =$ constant. The second is a condition of maximum in the sections of the surface at right angles to the former. If both of these conditions yield a maximum, the maximum point on the surface will be determined by combining the two equations. In this case the second condition does not yield a maximum, since when $i = 0$, w is zero for all values of E (235). The first condition has been already discussed.

129. *Power delivered to a motor.*

The power delivered to the motor is

$$W = Ei \text{.............................}(239).$$

This is also the equation of a surface. The first term of (235) represents the power W. That equation asserts that the power delivered by the motor is equal to the power delivered to it, less the power lost in the motor. The surface represented in (239) may be represented to the eye by filling in the space between the surface and its co-ordinate planes with plaster of Paris or other solid material. This surface is represented in Fig. 82. If either i or E be constant, the section of the surface is a right line, the equations of which are respectively

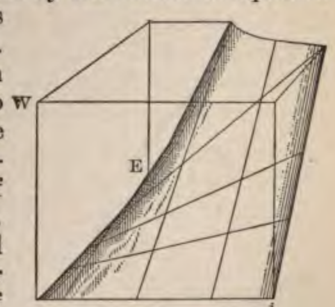


Fig. 82.

$$W = iE \text{ or } W = Ei.$$

If W be constant the section is an equilateral hyper-

bola referred to its asymptotes, which are in the planes W, E, and, W, i. Constructed in plaster as described the model will terminate upon the axes of E and i in an edge, shaped like a square chisel, and will overhang the surface w. The difference between the ordinates W and w for any point determined by E and i will be $R\,i^2$ and this difference represents the power lost in the motor itself.

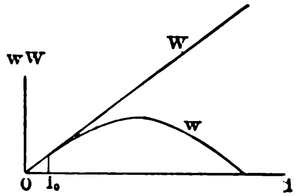


Fig. 83.

Any section of these two surfaces w and W at constant E is represented in Fig. 83.

The efficiency of the machine being represented by η its value is

$$\eta = \frac{w}{W} = \frac{Ei - Ri^2}{Ei}$$

$$= 1 - \frac{Ri}{E} \text{.................... (240).}$$

Since E is constant in the section of the surface we are considering, it is evident that we must here consider η as a function of i. For any given motor (which fixes R) supplied with electricity at any potential E, the efficiency will be perfect or unity when $i = 0$. This will occur when all of the load, including friction, is removed, and the motor is running at such a speed that its electromotive force E' which opposes E becomes equal to E. The efficiency thus becomes perfect when the output becomes zero. The efficiency becomes fifty per cent when $i = \frac{1}{2}\frac{E}{R}$ or when the output is a maximum, and it reduces to zero when $i = \frac{E}{R}$. There is no condition of maximum efficiency here. η diminishes steadily from 1 to zero as i increases from 0 to $\frac{E}{R}$, the rate of change being

$$\frac{d\eta}{di} = -\frac{R}{E}.$$

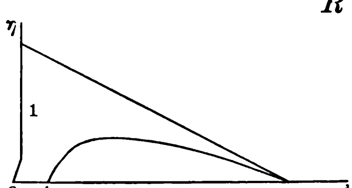


Fig. 84.

This is shown in the upper line in Fig. 84. Counting in the work lost in brush and journal friction as useful work, commercial motors may, therefore, be very efficient, but they must be allowed to run at high speed. The load must be much less than that corresponding to maximum output.

In Fig. 83, i_o may represent the current when the only load carried is that of brush and journal friction, or, in other words, when no useful work is being done. The commercial efficiency of the motor is zero under these conditions, while the electrical efficiency is very great, approaching unity. The commercial efficiency will, therefore, have a maximum, the conditions of which depend on the lubrication and finish of bearings, and the manner of applying the power. The lower curve in Fig. 84 shows in a general way how this efficiency is related to electrical efficiency.

Equation (228) shows that a condition of constant current i, is a condition of constant torque, since even in a self-exciting motor N would be constant if i is constant.

If i be made constant in (235) it becomes the equation of a straight line or

$$w = -Ri^2 + iE.$$

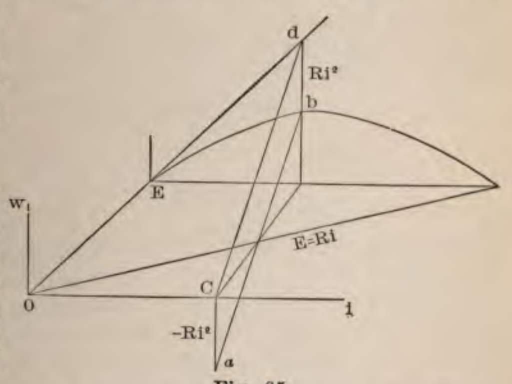


Fig. 85.

In Fig. 85 $a\ b$ represents such a line. It is an element of the surface w and has a negative intercept Ri^2. Such lines on the surface are, therefore, characterized by constant current and constant torque. The axis i is, therefore, also a torque axis, the relation between current and torque being given by (228). For self-exciting series motors the value of N is

nearly proportional to i for low degrees of saturation, and, therefore, T would be nearly proportional to i^2. In that case doubling the current would quadruple the torque, so that the torque scale is not a scale of equal divisions, when the lines of equal torque are drawn upon the surface w. It is evident that the line $c\,d$ of Fig. 85, which is parallel to $a\,b$, is an element of the surface W. This is equivalent to the statement that for a given current, the heat loss is constant for all values of E. This statement is not quite exact, when we include losses which take place in the iron core of an armature. These losses will be discussed later.

It may be stated before leaving this subject that the surface w and W are both hyperbolic paraboloids, but referred to different co-ordinate planes in space. There are two sets of rectilinear elements in each surface, by means of which the surfaces may be constructed with threads. In the surface W these lines are conditions of constant i and constant E, as is evident at once. In the surface w, we have just seen in Fig. 85 that constant i determines a linear section. The other linear section can be determined as follows.

Replacing w by its value $E'i$ in (235) and cancelling i, we have

$$E = E' + Ri \text{..................} (241).$$

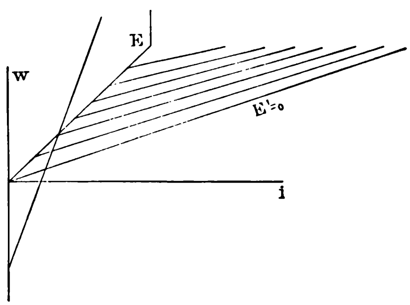


Fig. 86.

This for constant E' is the equation of a series of lines on the co-ordinate plane of E, i, beginning with $E' = 0$ when the motor is at rest. The lines for E'_1, E'_2 etc. will be parallel lines whose intercepts on the E axis are E'.

These lines are projections of the lines of constant E' on the surface w above. In order to find the equation of the projections on the plane w, E,

equation (241) must be combined with the equation of the surface (235) by the elimination of i. We thus obtain

$$w = -\frac{E'^2}{R} + \frac{E'}{R}E \qquad (242).$$

This is also the equation of a straight line, the negative intercepts on the w axis being $\frac{E'^2}{R}$. The slope of these lines being proportional to E', these projections are not parallel to each other. They, therefore, intersect each other. In order to determine the condition of this intersection, assume two equations (242) corresponding to E'_1 and E'_2. At the point of intersection they will have common values of E and w. Eliminating w in these equations and solving for E we have

$$E = E'_1 + E'_2 \qquad (243).$$

When the two projections have sensibly the same intercept E' on the E axis

$$E = 2E'.$$

Substituting this value of E' in (242) and reducing, we have as the equation of the curve to which these projections are tangent

$$w = \frac{1}{4}\frac{E^2}{R}.$$

This is identical with the equation of the projection of the line of maximum output (237). These considerations sufficiently determine the character of the surface w so that it may also be constructed by means of threads. Lines drawn on the surface w, representing constant current, and also those representing constant electromotive force of the motor, are straight lines in space.

130. *Determination of electromotive force in C. G. S. units.* The general method of determining potential difference in *C. G. S.* units may be understood from

the discussion of the simple case of a revolving circular current of one turn.

Let $Z\ N\ N'\ S$ be a circular closed wire of radius R' revolving around a vertical diameter, $Z\ N'$, in a field of magnetic force of strength H. The value of H has already been determined in $C.\ G.\ S.$ units (see sections 102–5). This field is the horizontal component of

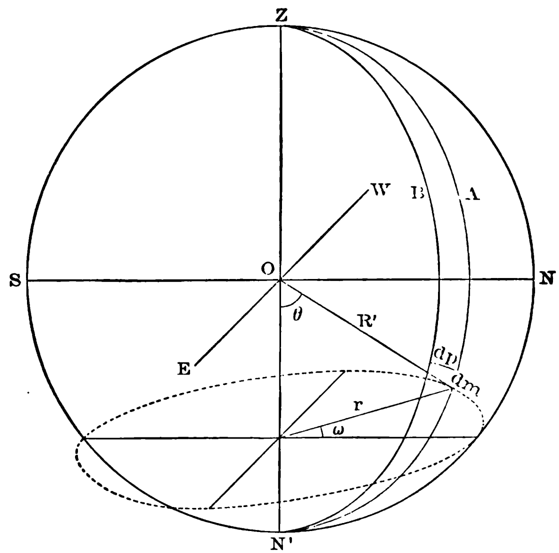


Fig. 87.

the earth's field, whose value in $C.\ G.\ S.$ units is about 0.20. Such a field would be represented by 20 lines of force to 100 square centimetres of area in a plane at right angles to the magnetic meridian N, S.

If the coil revolves at a uniform speed of n revolutions per second, it will be cut $4\ \pi\ R'^2\ H$ lines during every revolution, and $4\ \pi\ n\ R'^2 H$ lines during each second, but the number of lines cut per second will vary in a periodic manner during a revolution. This is due to the fact that equal meridian sectors of the surface of *revolution* are not transversed by an equal number of *lines.*

The rate of cutting lines will be greatest when the plane of the coil is in the magnetic meridian, and zero when it is 90° therefrom. The projection of the wire on the plane $Z\ W N'\ E$, at right angles to the magnetic meridian, will always cut the same number of lines per second as the wire, for while the distribution of lines over this plane is uniform, the velocity of the projection is not uniform, but periodic. We shall determine the number of lines cut per second by the wire projection at any instant. At any instant let $Z\ A\ N'$ represent the position of the coil the plane of which makes an angle ω with the meridian plane. During a time dt, the coil will sweep through an angle $d\ \omega$, the new position being $Z\ B\ N'$. The area of an element of the generated sphere will be

$$d^2S = dmdp$$

where dm is an element of one of the meridian circles, and dp is an element of the horizontal parallel at the same point, subtending the angle $d\ \omega$.

Let θ be the angle between the vertical radius $O\ N'$ and the radius R' drawn to the element d^2S, r being the radius of the horizontal parallel cutting d^2S.

Then

$$dp = rd\ \omega = R' \sin \theta d\omega$$
$$dm = R'\ d\ \theta$$
$$\therefore\ d^2S = dmdp = R'^2 \sin\theta d\theta d\omega.$$

We are now to find the area of the projection of d^2S on the plane $Z\ W\ N'\ E$, the projection being along the lines of force. If we project dm along the radius r upon the axis $Z\ N'$ and dm along the lines of force, the rectangle of thes wo proje tions will be equal in area to that of the projection of d^2S, since it will have the same base and an equal altitude.

Let d^2S', dm' and dp' be the projections of d^2S, dm and dp, then

$$dp' = dp \cos \omega$$
$$dm' = dm \sin \theta$$
$$d^2S' = dm'\ dp' = dmdp \sin \theta \cos \omega$$
$$= R'^2 \sin^2 \theta\ d\ \theta \cos \omega\ d\ \omega.$$

If we integrate this in θ between 0 and 2π we shall have the area of the projection of the spherical sector intercepted between the two planes, upon the plane of $Z\ W\ N'\ E$.

$$\text{Since} \int_0^{2\pi} \sin^2\theta\, d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \pi$$

$$\therefore\ dS' = \pi R'^2 \cos\omega\, d\omega.$$

The electromotive force being the number of lines cut per second, we have

$$E = H\frac{dS'}{dt} = \pi R'^2 H \cos\omega \frac{d\omega}{dt}.$$

If n be the number of rotations per second, the angular velocity of the wire is $2\pi n = \frac{d\omega}{dt}$. Also if A represents the area of the wire circle in square centimetres, $= \pi R'^2$ then

$$E = 2\pi n A H \cos\omega \ldots\ldots\ldots\ldots\ldots (244).$$

This equation determines E in any position during a revolution.

Since the value of the average cosine is $\frac{2}{\pi}$ the average value of E during a revolution is

$$E = 4 n A H \ldots\ldots\ldots\ldots\ldots\ldots\ldots (245).$$

We foresaw before beginning the discussion that this would be the average value of E.

If the coil be broken at N' and connected with commutator bars and brushes, so that this $E.\ M.\ F.$ generated by the coil can be balanced against that of a Daniell cell, it would be found that when the current is reduced to zero, so that a magnetic needle is not deflected, the value of E computed from the last formula is

$$E = 107{,}000{,}000.$$

In addition to H, this value (245) contains A, an area,

and n which is $\frac{1}{\text{a time}}$. This equation, therefore, determines the $E. M. F.$ of the cell in $C. G. S.$ units. In order to avoid writing, as cyphers, so many undetermined figures, a unit 10^8 times as large as the $C. G. S.$ unit has been adopted, and is known as the volt. The $E. M. F.$ of this cell is, therefore, 1.07 volts.

131. *Electromagnetic Determination of resistance in $C. G. S.$ units.*

The revolving coil of the previous section is used. A small spherical magnet is suspended at the middle of the coil upon a silk fibre which passes up through the hollow axis at Z. The magnet is deflected by the current generated in the revolving coil, and comes to rest when the turning moment due to the earth's field is balanced by that of the coil.

The force of the field on one pole of the magnet is Hm. The component of Hm resolved at right angles to the magnet is $Hm \sin a$ where a is the angle of deflection from the magnetic meridian. The effect of the coil on this pole is by (178)

$$f = \frac{2 \pi i m}{R'}$$

The value of i is by Ohm's law and Eq. (244)

$$i = \frac{E}{R} = \frac{2 \pi n A H}{R} \cos \omega$$

$$\therefore \quad f = \frac{4 \pi^2 A H m n}{R R'} \cos \omega \text{........ ....(246).}$$

This force is at right-angles to the coil, which makes an angle of $a - \omega$ with the needle. Therefore the component of f at right angles to the needle is

$$f' = \frac{4\pi^2 A H m n}{R R'} \cos \omega \cos (a - \omega) \text{.....(247).}$$

This force, therefore, fluctuates during a revolution. It is zero when $a - \omega = 90°$, or $270°$. The axis of the magnet is then at right angles to the coil. It is zero

when $\omega = 90°$ or $270°$. The plane of the coil is then at right angles to the magnetic meridian so that no lines are being cut. The sign of the turning moment is always the same, since reversals of current and of position accompany each other. Although the current fluctuates, the needle is perfectly steady under telescopic observation, when the speed is sufficiently great. The position of the needle is determined by the average value of f'. We must, therefore, determine the average value of the periodic function in (247). Expanding this expression

$$f(\omega) = \cos\omega \cos(a - \omega)$$
$$= \cos a \cos^2 \omega + \sin a \sin \omega \cos \omega.$$

To obtain the average of this expression it must be multiplied by $d\omega$, and integrated between 0 and $\frac{\pi}{2}$, dividing the integral by the corresponding arc $\frac{\pi}{2}$. In this way we have

$$\text{average} f(\omega) = \frac{2}{\pi}\left[\cos a \int_0^{\frac{\pi}{2}} \cos^2 \omega \, d\omega + \sin a \int_0^{\frac{\pi}{2}} \sin \omega \cos \omega \, d\omega\right]$$

$$= \frac{1}{2}\cos a + \frac{\sin a}{\pi}$$

$$\therefore \text{average} f' = \frac{4\pi^2 A H m n}{R R'}\left[\frac{1}{2}\cos a + \frac{\sin a}{\pi}\right].$$

For equilibrium of the needle this must equal $Hm \sin a$. Equating these values, and remembering that $A = \pi R'^2$ we have by solving for R,

$$R = 2\pi^2 n R' (2 + \pi \cot a).$$

The average distance of the elements of the wire from the axis of rotation is $\frac{2}{\pi} R'$, where $\frac{2}{\pi}$ is the average

sine. The average circumference described by these elements is, therefore, 4 R'. Hence the average velocity of the wire in space is

$$v = 4\,n\,R'.$$

Substituting this value of R' in the last equation we have

$$R = \frac{\pi^2}{2}\,(2 + \pi \cot \alpha)\,v \ldots\ldots\ldots\ldots\ldots (248).$$

Electrical resistance, therefore, has the dimensions of a velocity, since all other quantities in (248) are numerals. If a wire having a resistance of one ohm were to be thus experimented with, its resistance computed from the last equation would be 1,000,000,000 = 10^9, if v were expressed in centimetres per second. The ohm is, therefore, 10^9 *C. G. S.* units. A potential of V volts is $10^8 V$, in *C. G. S.* units, and a resistance of R ohms is $10^9 R$ in *C. G. S.* units. To avoid introducing a numeral factor into the equation for Ohm's law, the practical unit of current must be so chosen that i amperes must equal $10^{-1}\,i$ in *C. G. S.* units. Then

$$10^{-1} i = \frac{10^8\,E}{10^9\,R}.$$

In a current of one ampere, the quantity of electricity flowing per second is called the culomb. The culomb is 10^{-1} *C. G. S.* units of quantity.

132. *Heating effects of currents measured electrically in practical units.*

It is required to determine the heat generated per second in a wire of resistance R carrying a current i.

The rate of work in ergs per second required to force a current i (*C. G. S.*) through a resistance R (*C. G. S.*), the potential on the ends of the resistance, to which the flow is due being V (*C. G. S.*) is by (209)

$$\frac{W}{t} = i^2 R = V\,i = \frac{V^2}{R} = \frac{J\,H}{t} \ldots\ldots\ldots\ldots (249).$$

J represents the mechanical equivalent of heat, and H the number of heat units equivalent to W. H and W represent the work done upon or the heat developed in the resistance R in a time t.

One water-gramme-degree C of heat is by the best measurements found to be the equivalent of 426.4 gramme-metres (Rowland). This is equal to 42640 gramme-centimetres and $42640 \times 981 = 4.18 \times 10^7$ ergs. The latter quantity is, therefore, the value of J in ergs.

If i, R and V are to be expressed in amperes, ohms and volts, instead of in $C.\ G.\ S.$ units, these symbols must be replaced by $10^{-1}\,i$ $10^9\,R$ and $10^8\,V$ respectively. Therefore,

$$\frac{H}{t} = \frac{i^2R \times 10^{-2} \times 10^9}{4.18 \times 10^7} = i^2R \times 0.24$$

$$= \frac{V\,i \times 10^8 \times 10^{-1}}{4.18 \times 10^7} = V\,i \times 0.24$$

$$= \frac{V^2 \times 10^{16}}{R \times 10^9 \times 10^7 \times 4.18} = \frac{V^2}{R} \times 0.24.$$

An arc lamp having a potential difference of 30 volts on its terminals, and requiring a current of 10 amperes, the heat radiated per second from the arc would be $0.24\ V\,i = 300 \times 0.24 = 72$ calories. This would heat 72 grammes of water through $1°\ C$ each second.

133. *Rise in temperature of a wire carrying a current.*

A wire carries a current of i amperes. The resistance between two points on the wire is R ohms. The heat liberated in the wire between the two points will be $0.24\,i^2R$ calories. The resistance expressed in terms of the length, and radius of the wire, and the specific resistance K of the material of which the wire is conposed is $R = K\,\frac{l}{\pi r^2}$. K is the resistance in ohms of a conductor of the same material, one centimetre long and one square

centimetre in cross-section. For pure copper at 0°C, $K = 1.642 \times 10^{-6}$, and this value increases about 0.38 per cent. for each degree C. For the softest and purest wrought iron $K = 9.638 \times 10^{-6}$ at 0°C., and for grey cast iron $K = 1.056 \times 10^{-4}$. These coefficients increase about 0.365 and 0.083 per cent. respectively for one degree C. The heat generated in the wire each second is, therefore,

$$\frac{H}{t} = 0.24\ K \frac{li^2}{\pi r^2} \ldots\ldots\ldots\ldots(250).$$

The specific heat of the metal composing the wire being s, the density in grammes to the cubic centimetre being D, and the rise in temperature per second being T, the heat which the wire receives per second must also be

$$\frac{H}{t} = \pi r^2 l D s T \ldots\ldots\ldots\ldots(251).$$

Equating these equal values and solving for T we have

$$T = \frac{0.24\ K i^2}{\pi^2 D s r^4} = \frac{0.24\ K}{Ds} i^2_0$$

where

$$i_0 = \frac{i}{\pi r^2} \ldots\ldots\ldots\ldots(252)$$

or the current-density in amperes to the square centimetre of section.

A wire of pure copper being at a temperature of 20°C., or that of the surrounding air, the value of K would be 1.725×10^{-6}. $D = 8.9$ and $s = 0.0939$. If this wire were carrying 1000 amperes to the square inch, or 155 amperes to the square centimetre, the rise per second in the temperature would be

$$T = \frac{0.24 \times 1.725 \times 10^{-6} (155)^2}{8.9 \times 0.0939}$$
$$= 0.011 \text{ degree C.}$$

As the temperature of the wire rises, the heat will be radiated, the heat radiated per second being by Newton's

charge Q was stored in the condenser. The energy lost in the resistance R is, therefore, the same as if the energy per second converted into heat had decreased uniformly from $\frac{E^2}{R}$ to zero during a time-interval $R\ C$ Eq. (285).

The integral of the final term of (286) is the energy stored in the condenser. By Eq. (284) that term becomes

$$E\int_0^\infty idt - E\int_0^\infty i\,\varepsilon^{-\frac{t}{RC}}\,dt = \frac{E^2}{R}\int_0^\infty \varepsilon^{-\frac{t}{RC}}\,dt$$

$$-\frac{E^2}{R}\int_0^\infty \varepsilon^{-\frac{2t}{RC}}\,dt$$

$$= C\,E^2 - \frac{CE^2}{2} = \tfrac{1}{2}\,C\,E^2 \ldots\ldots\ldots\ldots\ldots\ldots(288),$$

which is identical with (287). The integral of $E\,i\,dt$ which represents the energy delivered by the source has been determined in the operation just finished, and its value is found to be $C\,E^2$. Hence whatever may be the resistance of a non-inductive conductor, connecting a condenser with a source of electricity having a fixed potential, half of the energy received by the conductor will be wasted in the conductor, and the other half will be stored in the condenser.

143. *Power during the charging of a condenser.*

Although the lost energy and the stored energy involved in the whole operation of charging a condenser are equal, and independent of the resistance, the rates at which energy is being lost and stored at any given instant are in general not equal.

Let H represent heat loss in watts and w the stored energy, then by (287) and (288), calling

$$dH = Ri^2dt \text{ and } dw = Eidt - Ei\,\varepsilon^{-\frac{t}{RC}}\,dt$$

An insulated wire wound into a coil has a diminished surface of radiation, which is that of the coil instead of that of the wire.

Newton's law of cooling should not be used for very large differences of temperature.

The hot-wire volt-meter is an application of the heating effects of a current. The elongation of a fine wire carrying a current may be utilized for measuring feeble currents, or potentials between 30 and 120 volts.

Let l = the length of the wire, r its radius and K the specific resistance of the material. Then the heat developed per second in the wire is by (250)

$$\frac{H}{t} = 0.24\ K \frac{i^2\, l}{\pi r^2}$$

The heat radiated per second is by (253)

$$\frac{H'}{t} = 2\, \pi\, h\, r\, l\, (T - T_0).$$

The elongation of the wire will cease when $H = H'$ or

$$i^2 = \frac{2\, \pi^2 h\, r^3}{0.24\ K} (T - T_0).$$

The elongation of the wire in being heated from T_0 to T is

$$l - l_0 = \alpha\, l_0\, (T - T_0)$$

where α is the coefficient of expansion of the metal.

$$\therefore\ i^2 = \frac{2\, \pi^2 h\, r^3}{0.24\ K\, \alpha\, l_0} (l - l_0).$$

We may also express r and K in terms of their values at 0° C. We have by section 18

$$r^3 = r^3_0\, (1 + \alpha\ T)^3 = r^3_0\, (1 + 3\ \alpha\ T).$$
$$K = K_0\, (1 + \beta\ T)$$

where β is the coefficient of increase in specific resistance.

$$\therefore\ i^2 = \frac{2\, \pi\, h\, (1 + 3\ \alpha\ T)\, r_0}{0.24\ \alpha\ K_0\, (1 + \beta\ T)}\, (l - l_0.)$$

$$= \frac{2\, \pi\, h\, r_0}{0.24\ \alpha\ K_0}\ [1 + (3\ \alpha - \beta)\ T]\, (l - l_0).$$

tial during the period. The voltage varies according to the equation

$$E = E_0 \sin 2\pi \frac{t}{T}.$$

$$\therefore E_0 = \sqrt{2} \times \text{virtual voltage.}$$

134. *Horse-power measured electrically.*

A resistance R ohms having a current i amperes will receive and radiate 0.24 $R i^2$ heat units per second. The equivalent number of ergs per second is $R\, i^2 \times 10^7$ (section 132). For convenience in practical measurements 10^7 ergs per second is taken as a practical power unit, and is called a watt. The power applied to the above conductor is, therefore, $R\, i^2$ watts, where R and i are in ohms and amperes respectively.

One horse-power is a rate of work of 76 kilogramme-metres per second. This is equal to 7.6×10^6 gramme-centimetres or $7.6 \times 10^6 \times 981$ dyne-centimetres, or ergs, per second, or 746×10^7. Therefore one horse-power is equal to 746 watts. It follows that the horse-power applied to a conductor having a resistance R in ohms, carrying a current of i amperes is

$$H.\ P. = \frac{Ri^2}{746} = \frac{Vi}{746} = \frac{V^2}{746\, R} \text{.........} \quad (255).$$

The arc lamp referred to at the close of section 132, which radiates 72 calories per second would, therefore, consume energy at the rate of $\frac{30 \times 10}{746} = 0.40$ horse-power.

135. *Heat developed in a net-work of conductors.*

In any divided circuit whose branches have resistances r_1 and r_2 carrying currents i_1 and i_2, we know experimentally that the current in the undivided circuit is

$$i = i_1 + i_2 \text{.....................................} \quad (256).$$

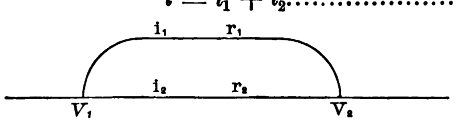


Fig. 88.

The power in watts, radiated as heat from the two branches, is

$$w = i_1^2 r_1 + i_2^2 r_2 \text{.....................................} (257).$$

Suppose the current i to divide arbitrarily between the branches, but still satisfying (256), so that any increment added to i_1 must be taken from i_2. This might come about from a design to relieve one overburdened and overheated wire of part of its current and shift part of the load to the cooler wire, in the interests of a more equitable arrangement. The condition imposed is then obtained by differentiating (256) or

$$0 = d\,i_1 + di_2 \qquad \text{(258)}.$$

It is then evident that such arbitrary redistribution of current between the branches would affect the total heat w, generated in the whole system r_1 and r_2. The condition that this total heat shall be a minimum is found by differentiating (257), and replacing di_2 by its value $-di_1$. In this way we have

$$\frac{dw}{di_1} = 2\,i_1\,r_1 - 2i_2\,r_2 = 0$$

$$\frac{i_1}{i_2} = \frac{r_2}{r_1}.$$

The currents must divide between the two branches, inversely as the resistances in those branches. Experiment shows that these currents do actually divide in this manner. This is expressed in Ohm's law. The above discussion shows that if the distribution were different, the total heat of the system would be greater than it now is.

In order to apply this discussion to a more general case we must determine the single resistance equivalent to the two resistances r_1 and r_2 in multiple.

Let $V_2 - V_1$ be the difference in potential between the two points where the current divides. Then by Ohm's law

$$V_2 - V^1 = i_1\,r_1 = i_2\,r_2$$

$$= \frac{i_1}{\frac{1}{r_1}} = \frac{i_2}{\frac{1}{r_2}} = \frac{i_1 + i_2}{\frac{1}{r_1} + \frac{1}{r_2}} = \frac{i}{\frac{1}{r_1} + \frac{1}{r_2}} \qquad \text{(259)}.$$

A single resistance R' replacing the two resistances

without affecting the external current i must satisfy the equation

$$V_2 - V_1 = i\,R' = \frac{i}{\frac{1}{R'}} \qquad (260).$$

From (259) and (260) it is evident that

$$\frac{1}{R'} = \frac{1}{r_1} + \frac{1}{r_2}$$

$$\text{or } R' = \frac{r_1\,r_2}{r_1 + r_2} \qquad (261).$$

If $r_1 = r_2 = r$ then $R' = \frac{r}{2}$.

If $r_1 = 10$ how will R' vary if r_2 vary from zero to an infinite value? We have

$$R' = \frac{10\,r_2}{10 + r_2} \qquad (262).$$

It is evident that if r_2 becomes very small, the equation becomes

$$R' = \frac{10\,r_2}{10} = r_2$$

or the equivalent resistance is equal to the smaller resistance, and approaches zero as it approaches zero.

If r_2 becomes very large the equation becomes

$$R' = \frac{10\,r_2}{r_2} = 10$$

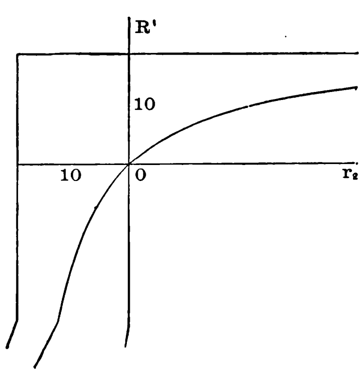


Fig. 89.

or the equivalent resistance is again equal to the smaller.

Equation (262) is the equation of an equilateral hyperbola shown in Fig. 89. For instance a volt-meter having a resistance of 15,000 ohms is connected on the terminals of an incandescent lamp having a terminal potential difference of 55 volts, and requiring a current of one ampere. This would be a 55 watt

lamp. Its resistance would be 55 ohms. By connecting the voltmeter in multiple with the lamp, the resistance between the points of connection becomes

$$\frac{55 \times 15{,}000}{55 + 15{,}000} = 54.79 \text{ instead of } 55.$$

Let a third branch having a resistance r_3 and a current i_3 be connected in multiple with the two former branches, which also have resistances a and b in series with them, as shown in Fig. 90. We have already determined the law for the distribution of current i'' in a or b between r_1 and r_2. The resistance of the compound circuit between the points where circuit r_3 is connected is

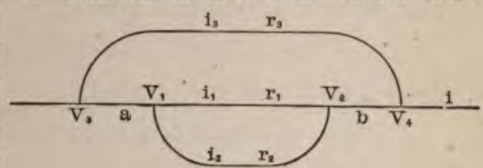


Fig. 90.

$$r' = a + \frac{r_1 r_2}{r_1 + r_2} + b$$

We may, therefore, consider this branch replaced by a single branch r' which can be determined if a, b, r_1 and r_2, are known. The resistance of the entire system will then be

$$R' = \frac{r_3 r'}{r_3 + r'}.$$

and the division of current between r_3 and r' in order that the heat in the entire system may be a minimum is

$$\frac{i_3}{i'} = \frac{r'}{r}.$$

In this manner the discussion may be extended, to include any number of branches connected in any specified manner.

Example: Suppose in a divided circuit Fig. 88 that $r_1 = 3$ ohms, $r_2 = 9$ ohms, and let the current in the undivided branch be $i = 50$ amperes.

Then by (257)

$$w = 3\, i_1^2 + 9\, i_2^2.$$

By (256)

$$50 = i_1 + i_2.$$

Let us assume that $i_1 = n \times 50$ where n may be any fraction between zero and unity. Then

$$i_1 = 50\,n$$
$$i_2 = 50\,(1 - n)$$
$$\therefore\ w = 2500\,(3\,n^2 + 9\,(1 - n)^2)\dots\dots\dots(263).$$

When $n = 0$, $i_1 = 0$ and $i_2 = 50$

$$w = 2500 \times 9 = 22{,}500 \text{ watts}.$$

When $n = 1$, $i_2 = 0$ and $i_1 = 50$.

$$w = 2500 \times 3 = 7500 \text{ watts}.$$

The condition that will make w a minimum is $n = \frac{3}{4}$, in which case $w = 5625$.

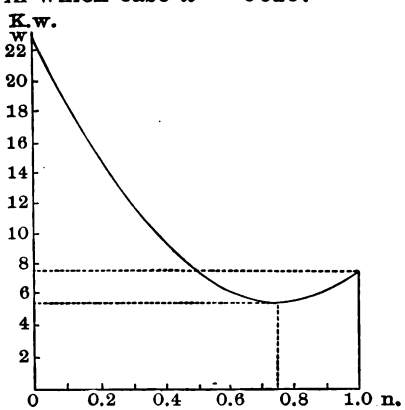


Fig. 91.

The manner in which w varies with variations in n is shown in Fig. 91 where w is given in kilowatts. This curve, of which (263) is the equation, is a parabola. The curve extends beyond the limits computed for it here, but outside of those limits the imposed physical conditions are not satisfied by the equation, or its curve. It will be observed that if all of the current were to be shifted to the wire which carries the greater current when Ohm's law is satisfied, the resulting heat in the system would be less than if it were to be all thrown into the wire which carries the smaller current.

If we solve (263) for n we have

$$n = \frac{3}{4} \pm \sqrt{\frac{w}{30{,}000} - \frac{3}{16}}$$

This equation shows that the line of symmetry of the curve is

$$n = \frac{3}{4}$$

which corresponds to the condition of minimum. When the $\pm$ term becomes zero, $w = 5625$, which also corresponds to the condition of minimum. All values of w less than this give imaginary results.

CHAPTER VIII.

VARIABLE CURRENTS, WITH E. M. F. FROM CONSTANT SOURCE.

136. *Current on closing a circuit with fixed E. M. F.* Equation (213) may be written:

$$dt = \frac{L}{R} \frac{di}{\frac{E}{R} - i} \quad \text{.............................. (264).}$$

Integrating this equation between the limits $t = 0$, $i = 0$ and t, i, we can determine what the current will be t seconds after the circuit has been closed. We thus have

$$t = \frac{L}{R} \int_{o}^{i} \frac{di}{\frac{E}{R} - i} = -\frac{L}{R} \log_{\varepsilon} \frac{\frac{E}{R} - i}{\frac{E}{R}}$$

$$\therefore -\frac{Rt}{L} = \log_{\varepsilon} \frac{\frac{E}{R} - i}{\frac{E}{R}}$$

and

$$i = \frac{E}{R} - \frac{E}{R} \varepsilon^{-\frac{Rt}{L}} \quad \text{...............(265).}$$

where ε is the Naperian base.

This equation shows that when $t = 0$, $i = 0$. As t increases, i increases, and after an infinite time $i = \frac{E}{R}$.

Practically, however, the exponent $\frac{Rt}{L}$ is, or may be,

very large when t is a small fraction of a second, and, therefore, the second term of the final member may vanish very quickly. This is due to the fact that when measured in their appropriate units, the resistance of any ordinary circuit is very large compared with its coefficient of self-induction L.

The larger L is the more slowly will the current rise to a value which is practically $\frac{E}{R}$. A circuit containing an enormous electro-magnet or a great number of magnets, or a circuit having the primaries of a great number of transformers, would have a large self-induction coefficient L, and the current would rise much more slowly, than in one having only a battery and a connect-

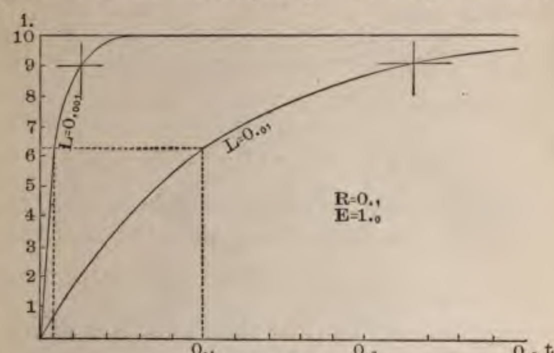


Fig. 92.

ing wire. At the surface of the wire in any part of the circuit the magnetic force is $\frac{2i}{r}$, where r is the wire radius. (Section 116.) As i increases, the force at the wire surface increases, and a line of force which at one instant has a radius r, at the next instant will have a much larger radius. The lines of force due to each of the windings of a magnet coil thus crowd outwards from the wires in which they originate, and cutting across the neighboring wires find their way into the iron core which serves as a conductor for such lines.

This results in at once establishing an opposing *E. M. F.* which has been already discussed in section 123. If $L = 0$ Eq. (265) represents simply Ohm's law. Fig. 92 shows two curves which illustrate the rise of a current in a circuit where the constant *E. M. F.* is one volt, the resistance 0.1 ohm and where the final current after an infinite time is, therefore, 10 amperes. In the lower curve the coefficient L is 0.01 and in the upper it is 0.001 henry.

These curves are computed from the equation

$$log_{10}\left(\frac{E}{R} - i\right) = log_{10}\frac{E}{R} - M\frac{R}{L}t$$

where M is 0.4343, the modulus of common logarithms. For the lower curve

$$log\,(10 - i) = 1 - 4.343\,t$$

and for the upper

$$log\,(10 - i) = 1 - 43.43\,t.$$

In the circuit having the greater self-induction, the current will rise to one ampere less than the final value in a time $t = \dfrac{1}{4.343}$ second $= 0.230$. The current will become 9 amperes in a little less than a quarter of a second. The origin of this logarithmic curve is, therefore, at a point whose co-ordinates are $t = 0.23$, $i = 10$. The vertical line through this point is the axis of numbers, and the horizontal line is the axis of logarithms. In the other circuit the current will rise to 9 amperes in a time $t = \dfrac{1}{43.43} = 0.0230$ second. If $t = \dfrac{L}{R}$ in (265) that equation becomes

$$i = \frac{E}{R} - \frac{1}{\varepsilon}\frac{E}{R}$$

or the current is less than its final value by $\dfrac{1}{\varepsilon}$th part of *that value.* This value of t is called the time constant

of the circuit. In the lower curve the time constant is 0.1 and in the upper curve it is 0.01 second.

137. *Energy required in exciting a magnet.*

The energy applied to the circuit by the battery during the time the current is increasing, is by (213) $\int_0^\infty E\, i\, dt$ where i is given by (265). The heat measured in watt-seconds, generated in the circuit during the same time is by (213) $\int_0^\infty R\, i^2 dt$ and the energy stored up in the field is $\int_0^\infty L\, i\, d\, i$. We shall examine each of these terms. We have first

$$\int_0^\infty E\, i\, dt = \frac{E^2}{R}\int_0^\infty dt - \frac{E^2}{R}\int_0^\infty \varepsilon^{-\frac{Rt}{L}}\, dt.$$

$$= \frac{E^2}{R}\int_0^\infty dt + \frac{E^2}{R}\left[\frac{L}{R}\,\varepsilon^{-\frac{Rt}{L}}\right]_0^\infty$$

$$= \frac{E^2}{R}\int_0^\infty dt - \frac{E^2}{R}\frac{L}{R} \qquad \text{(266).}$$

The value of the first term of (266) is evidently infinite.

If it takes an infinite time for the current to rise to its limiting value $\frac{E}{R}$, and if in a few hundredths of a second it practically reaches its limiting strength, the heat delivered by the battery during the total rise must necessarily be infinite.

For the energy in watt-seconds received by the circuit in the form of heat, we have (265)

$$\int_0^\infty Ri^2 dt = \frac{E^2}{R}\int_0^\infty dt - 2\frac{E^2}{R}\int_0^\infty \varepsilon^{-\frac{Rt}{L}} dt + \frac{E^2}{R}\int_0^\infty \varepsilon^{-\frac{2Rt}{L}} dt$$

$$= \frac{E^2}{R}\int_0^\infty dt - 2\frac{E^2}{R}\frac{L}{R} + \frac{E^2}{R}\frac{L}{2R}$$

$$= \frac{E^2}{R}\int_0^\infty dt - \frac{3}{2}\frac{E^2}{R}\frac{L}{R} \text{............} (267).$$

For the energy stored in the field of the circuit during the operation, we have since

$$di = \frac{E}{L}\varepsilon^{-\frac{Rt}{L}} dt$$

$$\int_0^\infty L\, i\, di = \frac{E^2}{R}\int_0^\infty \varepsilon^{-\frac{Rt}{L}} dt - \frac{E^2}{R}\int_0^\infty \varepsilon^{-\frac{2Rt}{L}} dt$$

$$= \frac{E^2}{R}\frac{L}{R} - \frac{E^2}{R}\frac{L}{2R}$$

$$= \frac{1}{2}\frac{E^2}{R}\frac{L}{R} = \frac{1}{2}L\, i_0^2 \text{...............} (268).$$

$$\text{where } i_0 = \frac{E}{R}.$$

This result is the difference between the second members of (266) and (267) the infinite terms of those members disappearing

The last equation shows that the energy stored in the field of the circuit is the same as if the energy were *being* stored at a uniformly increasing rate, beginning

with zero, and ending with $\frac{E^2}{R}$ during a time interval $\frac{L}{R}$, which is the time constant of the circuit.

In the two circuits represented in Fig. 92, the energy stored in the field is

1. $L = 0.01$; $\frac{E^2}{R^2} = 100$; energy $= \frac{1}{2}$ watt-second.

2. $L = 0.001$; $\frac{E^2}{R^2} = 100$; energy $= \frac{1}{20}$ watt-second.

The watt-second is called a joule, and is 10^7 ergs.

We may determine what must be the coefficient of self-induction in an electromagnet or motionless dynamo in order that it may require n horse-power-seconds or $n \times 746$ watt-seconds or $n \times 746 \times 10^7$ ergs to bring its field up to any given condition. Suppose the dynamo or magnet to be put in circuit with a battery having $E = 100$ volts. Let the final current be 50 amperes. The resistance of the entire circuit is then 2 ohms. Then by (268)

$$746\, n = \frac{1}{2}\left(\frac{E}{R}\right)^2 L$$

$$L = \frac{1492}{2500}\, n.$$

If $n = 10$, $L = 5.97$ henrys.

An electro-magnet having this coefficient of self-induction, and traversed by a current of 50 amperes, has stored up in its field an amount of energy equal to 7.46×10^{10} ergs. This energy would lift a metric ton, or 1,000 kilogrammes (2200 lbs.) through a height of about 76 centimetres. The entire expression in equation (268) represents energy. Since $\frac{E^2}{R}$ is an expression for power, and represents the power finally applied after the current becomes constant, it is clear that $\frac{L}{R}$ must have the dimensions of a time-interval the unit of which is the

second. We have already found that R has the dimensions of a velocity. It follows that L must have the dimensions of a length. The practical unit in which L, the coefficient of self-induction is measured, is the henry. It corresponds to the volt and the ohm. As we shall see subsequently, the henry is 10^9 C. G. S. units. It follows that the change from C. G. S. units to practical units may be considered as involving a change in the unit of length from the centimetre to 10^9 centimetres or a quadrant of the earth.

That the coefficient of self-induction in any coil is small as compared with its resistance may be seen from the following considerations. Suppose the radius of a circular coil of one turn to be 100 centimetres and the radius of the wire to be one centimetre. The length of the wire would then be 628 centimetres, $= 2\,\pi\,100$ and its cross section would be π square centimetres. If the resistance of one centimetre of the wire be taken as 0.000,002 ohm, the resistance of the entire circle will be

$$\frac{2\,\pi\,100}{\pi} \times 0.000{,}002 = 0.0004 \text{ ohm} = 400000\ C.\ G.\ S.$$

units. By reference to sections 115–6 it will be seen that the number of lines due to a *C. G. S.* unit current in the wire, which link with the wire is, Eq. (197) $2\,\pi^2 \times 100 \times (0.99)^2 \times 2.985 = 5775$ external and 628 internal. The total number will be 6403. This is the value of L for this wire. The value of $\frac{R}{L}$ is, therefore 62.4. If the coil be composed of two windings it will double this resistance, it will double the number of lines due to the coil and it would double the number of times each line links with the circuit. This would quadruple the value of L and the ratio $\frac{R}{L}$ would be half as great. This would not hold for widely separated windings of a coil, since the lines due to one winding would not all link with the other.

138. Power required at any instant to excite a magnet. We have already determined in (266) the time integral

of the power $E\,i$ during this operation. Calling $Eidt = d\,w$ we have for the power (265)

$$\frac{dw}{dt} = Ei = \frac{E^2}{R} - \frac{E^2}{R}\,\varepsilon^{-\frac{Rt}{L}} \text{.........(269).}$$

We may apply this equation to the data represented in the lower curve in Fig. 92, where $R = 0.1$ ohm, $E = 1$ volt, and $L = 0.01$ henry. Inserting these values equation (269) becomes,

$$\frac{dw}{dt} = 10 - 10\,\varepsilon^{-10\,t} \text{......................(270).}$$

Taking Naperian logarithms

$$log_\varepsilon\left(10 - \frac{dw}{dt}\right) = log_\varepsilon\, 10 - 10\,t.$$

In common logarithms

$$log\left(10 - \frac{dw}{dt}\right) = 1 - 4.3429\,t.$$

From which $\frac{dw}{dt}$ can be computed for any value of t.

When $t = 0$ it is evident that $\frac{dw}{dt} = 0$. If $t = 0.5$

$$log\left(10 - \frac{dw}{dt}\right) = 1 - 2.1724 = 8.8276$$

$$\therefore\quad 10 - \frac{dw}{dt} = 0.067$$

$$\frac{dw}{dt} = 9.93 \text{ watts.}$$

In this way are computed the various points on the logarithmic curve, whose equation is (270). This curve is shown in Fig. 93. It is the uppermost curve.

The power applied approaches a limit of ten watts, which becomes the power when t is infinite.

We proceed to determine how this power is distributed between heat lost in the copper, and energy stored in the

magnetic field during the operation. Calling $d\ H$ the lost heat or $d\,H = R\,i^2\,dt$ we have from (265)

$$\frac{dH}{dt} = \frac{E^2}{R} - 2\frac{E^2}{R}\,\varepsilon^{-\frac{Rt}{L}} + \frac{E^2}{R}\,\varepsilon^{-\frac{2Rt}{L}} \quad \text{.......} (271).$$

This gives the lost power in watts when E and R are in volts and ohms, and L and t are in henrys and seconds.

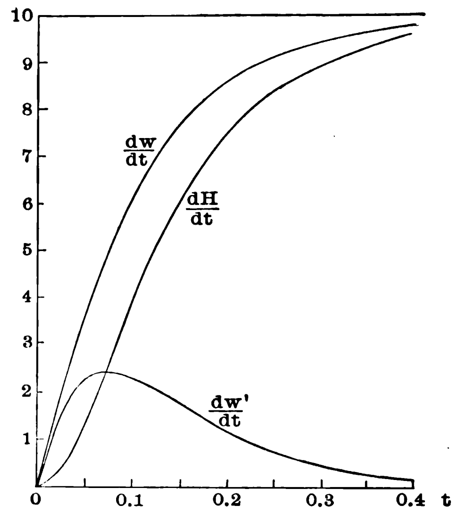


Fig. 93.

$\frac{dw}{dt}$ represents the rate at which energy is being used in exciting a magnet. $\frac{dH}{dt}$ shows the rate at which this energy is being converted into heat. $\frac{dw'}{dt}$ shows the rate at which this energy is being stored in the magnetic field.

Inserting the numerical values used in the previous computation the last equation becomes

$$\frac{dH}{dt} = 10 - 20\varepsilon^{-10t} + 10\,\varepsilon^{-20t} \quad \text{...........} (272).$$

The first term, 10, is constant for all values of t, and is,

therefore represented in Fig. 93 by the horizontal line at the top of the figure. The second term is represented by a logarithmic curve like that previously computed, as representing $\frac{dw}{dt}$. It may be computed and plotted as before described. Its ordinates are all negative, the initial ordinate when $t = 0$ being -20. The third term is represented by a similar logarithmic curve whose ordinates are positive. The initial ordinate when $t = 0$ is 10. The algebraic sum of the ordinates of these lines is represented by the curve marked $\frac{dH}{dt}$ in Fig. 93. The area of this curve is the total energy dissipated as heat during the infinite time required to excite the magnet. It is the integral determined in Eq. (267). Its value is also evidently infinite.

For the power stored in the magnetic field we have calling $dw' = Lidi$ (265) and (268)

$$\frac{dw'}{dt} = \frac{E^2}{R}\varepsilon^{-\frac{Rt}{L}} - \frac{E^2}{R}\varepsilon^{-\frac{2Rt}{L}} \qquad \text{.........(273).}$$

Inserting the same numerical values used in computing the other two curves of Fig. 93.

$$\frac{dw'}{dt} = 10\varepsilon^{-10t} - 10\varepsilon^{-20t} \qquad \text{...............(274).}$$

The terms of this equation can be computed as above explained. The ordinates of the resulting curve are represented by the lower curve in Fig. 93. Its ordinates are evidently equal to the difference between the ordinates of the other two curves.

The rate at which energy is being used to create a magnetic field, whose constantly increasing lines link with the constantly increasing current to which the lines are due, is shown by the ordinates of this curve. After a fraction of a second, the condition which will be reached in an infinite time has been practically reached, and the rate at which energy is being used in producing the

magnetic field becomes very small. What Faraday called the electrotonic state has been practically reached.

From equation (273) we may determine the time when the power used in creating the system is a maximum. We have

$$\frac{d^2 w'}{dt^2} = -\frac{E^2}{R}\frac{R}{L}\varepsilon^{-\frac{Rt}{L}} + \frac{E^2}{R}\frac{2R}{L}\varepsilon^{-\frac{2Rt}{L}} = 0$$

or

$$t_m = \frac{L}{R}\log_\varepsilon 2 = 0.693\frac{L}{R} \quad \ldots\ldots\ldots\ldots(275).$$

where $0.693 = \frac{\log_{10} 2}{M}$, M being the modulus of common logarithms. At this time the power stored and wasted are equal. In the particular case represented in Fig. 93 the time of maximum is $t = 0.069$. The area of this curve is the energy stored during the operation, the value of which is represented in Eq. (268).

139. *Discharge of a Magnet.*

If when the current shall have become practically constant, the source of *E. M. F.* be stricken from the circuit, R remaining constant, the condition of magnetic stress in the field cannot be maintained. The energy $\frac{1}{2} L i^2$ flows back upon the wire. The lines of magnetic force which link with the windings of the circuit diminish in area, and disappear in the windings of the magnet, in which they originated. The equation of energy (213) then becomes

$$-L\,i\,di = Ri^2\,dt$$

since the sign of di becomes negative. From this equation

$$-\frac{Rt}{L} = \int_{\frac{E}{R}}^{i}\frac{di}{i} = \log_\varepsilon \frac{i}{\frac{E}{R}}.$$

$$\therefore i = \frac{E}{R} \varepsilon^{-\frac{Rt}{L}} \qquad \text{.........................(276).}$$

This is the equation of the same logarithmic curve as that represented by equation (265) but referred to a different origin. The time rate of increase in the one case, is the same as the time rate of decrease in the other, being without regard to sign

$$\frac{di}{dt} = \frac{E}{L} \varepsilon^{-\frac{Rt}{L}} \qquad \text{..........................(277).}$$

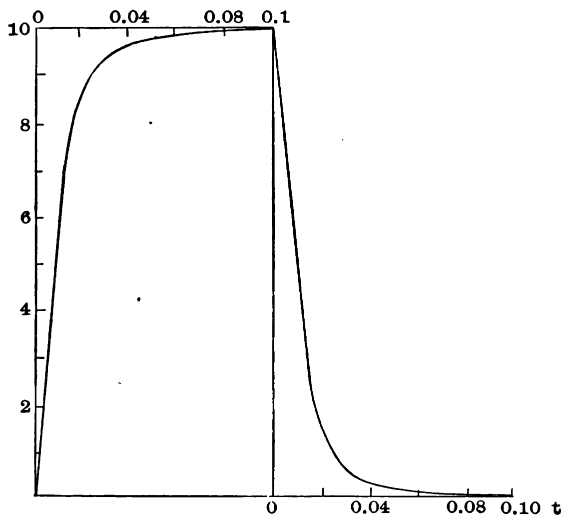


Fig. 94.
Currents during charge and discharge of a magnet.

Fig. 94 shows one of the curves of Fig. 92 and the current of discharge obtained on opening the circuit 0.1 second after it was closed.

It will be seen that the direction of the current of discharge is the same as that of the current during charging or exciting the magnet, although the *E. M. F.* of self-induction is oppositely directed. The current in exciting the magnet is due to the external source, and

the *E. M. F.* of self-induction opposes it. If L were zero, the current would rise abruptly to $\frac{E}{R}$. The current of discharge is due to self-induction. If L were zero, i would then be zero for all values of t greater than $t = 0$. The current would sink abruptly to zero on opening the circuit.

140. *Energy and power during the discharge of a magnet.*

Remembering that the sign of di is negative during discharge, the energy impressed upon the circuit during discharge is

$$-\int_0^\infty L\, i\, di = \frac{E^2}{R}\int_0^\infty \varepsilon^{-\frac{2Rt}{L}}\, dt.$$

where the value of i and $d\,i$ are obtained from (276). By integration, calling w the energy of discharge

$$w = -\frac{E^2}{R}\frac{L}{2R}\left[\varepsilon^{-\frac{2\,Rt}{L}}\right]_0^\infty$$

$$= \frac{1}{2}\;\frac{E^2}{R}\frac{L}{R} = \frac{1}{2}\;\mathrm{L}\, i^2_0. \qquad \text{...............(278)},$$

which is the same as the energy stored in the magnet during charging (268).

The energy is, however, restored at a very different rate from that at which it was stored during the operation of charging.

Calling dw the energy restored to the circuit in a time dt, we have

$$\frac{dw}{dt} = \frac{-L\, idi}{dt} = \frac{E^2}{R}\;\varepsilon^{-\frac{2Rt}{L}} \qquad \text{.......(279)}.$$

Substituting the values previously used in Fig. 93,

$$\frac{\mathrm{dw}}{\mathrm{dt}} = 10\,\varepsilon^{-20\,t} \qquad \text{... .... (280)}.$$

From this equation the ordinates for the curve in Fig. 95 have been computed. This curve shows the power restored to the circuit by the discharging magnet, at each instant during the discharge. The curve $\frac{dw'}{dt}$ in the same figure is reproduced from Fig. 93, and shows the power stored in the same magnet during the operation of charging. The areas of these two curves represent the

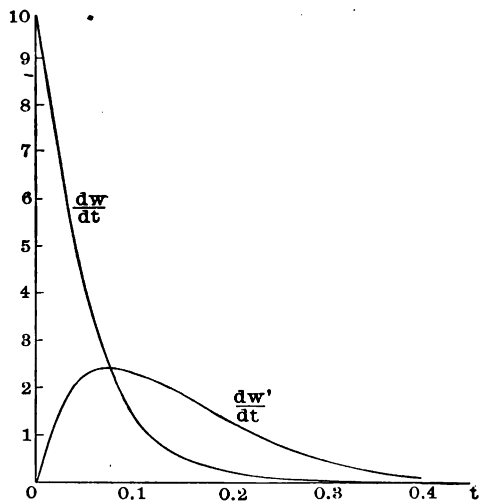


Fig. 95.

$\frac{dw}{dt}$ shows the rate at which energy is restored from the magnetic field and appears as heat during the discharge of a magnet. $\frac{dw'}{dt}$ shows the rate at which energy is stored in the magnetic field during charging.

total energy determined in equations (278) and (268) and are equal. The curves cross at the point of maximum in the curve $\frac{dw'}{dt}$ as is easily shown by equating (279) and (273) which will give (275).

The energy of this discharge is expended in heating

the wires, under the conditions supposed. This energy will be (276)

$$\int_0^{\infty} Ri^2\,dt = R\frac{E^2}{R^2}\int_0^{\infty}\varepsilon^{-\frac{2Rt}{L}}\,dt$$

$$= \frac{1}{2}\frac{E^2}{R}\frac{L}{R} = \frac{1}{2}L\,i_0$$

which is identical with (278). The power applied to the heating of the wire at any instant is by (276)

$$\frac{Ri^2\,dt}{dt} = \frac{dw}{dt} = R\frac{E^2}{R^2}\varepsilon^{-\frac{2Rt}{L}}$$

which is identical with (279). The curve marked $\frac{dw}{dt}$ in Fig. 95, therefore, represents the rate at which energy is restored to the circuit during discharge of a magnet, and also the rate at which power is lost in heating the circuit during the same operation.

141. *Current on closing a circuit having a non-inductive resistance and a condenser.*

Let C be the capacity of the condenser. The potential difference of the condenser terminals at any instant will be $\frac{Q}{C}$. The current flowing into the condenser will be $\frac{dQ}{dt}$. During a time dt, the energy stored in the condenser will, therefore, be $\frac{Q}{C}\frac{dQ}{dt}dt$. The energy equation (213) therefore becomes

$$E\,i\,dt = R\,i^2\,dt + \frac{Q}{C}\frac{dQ}{dt}\,dt \text{ ....(281)},$$

where R is the entire resistance of the circuit. If we suppose that the condenser is connected to constant potential mains, having a difference of potential E, then R will be the resistance through which current is fed from such mains to the condenser.

The potential difference of the plates of the condenser being $\frac{Q}{C}$, this potential will oppose E. Replacing i by $\frac{dQ}{dt}$ in (281) and cancelling the common factors, we have,

$$E = R\frac{dQ}{dt} + \frac{Q}{C} \text{ ....} (282).$$

$$\therefore\ dt = RC\frac{dQ}{EC - Q}$$

Counting t from the instant of closing the circuit when $Q = 0$ we have by integration,

$$t = RC \int_0^Q \frac{dQ}{EC - Q} = -RC \log_\varepsilon \frac{EC - Q}{EC}$$

The total charge in the condenser at any time t is found by solving this expression for Q, or

$$Q = CE - CE\varepsilon^{-\frac{t}{RC}} \text{ ....} (283).$$

But $E\ C$ is the final charge of the condenser, after the difference of potential on its plates becomes E, or that of the source. The unit of capacity corresponding to the ohm, volt, and ampere, is the farad and is 10^9 $C.\ G.\ S.$ electromagnetic units. A condenser of one farad capacity would require a charge of one culomb to raise the potential of one terminal to one volt, if the other be grounded. As we shall see further along, the capacity of the entire earth is only 0 0007 farad, so that capacities ordinarily existing in wire systems are very small. It folows that when the resistance of a circuit is a few ohms only, the exponent $\frac{t}{RC}$ of equation (283) is very large. According to that equation t must be infinite before the condenser will be charged to its final potential so that $Q = C\ E$. But since C, and also $R\ C$, is very small,

when a battery is connected with condensers of ordinary capacity by metal wires, the charge practically reaches its limiting value in a very small fraction of a second.

A condenser of 5 micro-farads or 5×10^{-6} farads has a potential of 2,000 volts suddenly applied to it through a non-inductive resistance of 200 ohms.

Then $C\,E = 10{,}000 \times 10^{-6} = 0\;01$ culomb. This will be the final charge, after an infinite time.

$$R\,C = 200 \times 5 \times 10^{-6} = 0.001.$$

Writing (283) in different form we have

$$C\,E - Q = C\,E\,\varepsilon^{-\frac{t}{RC}}$$

$$log_{10}\,(C\,E - Q) = log_{10}\,C\,E - \frac{Mt}{RC}$$

where $M = 0.4343$, the modulus of common logs. The equation then becomes

$$log_{10}\,(0.01 - Q) = -2 - 434.3\,t.$$

If $t = 0.001$ second

$$log\,(0.01 - Q) = -2 - 1 + 0.5657$$
$$0.01 - Q = 0.00368$$
$$Q = 0.0063 \text{ culomb.}$$

Fig. 97 shows how the quantity in the condenser increases. Practically the limiting charge is reached in about 6 or 7 thousandths of a second. The current at any instant being $i = \frac{dQ}{dt}$, it may easily be found by differentiating (283). In this way we have,

$$i = \frac{E}{R}\,\varepsilon^{-\frac{t}{RC}} \;\ldots\ldots\ldots\ldots\ldots\ldots\; (284).$$

By reference to these two equations it will be observed that when $t = 0$ and Q is, therefore, zero, the current is wholly independent of the capacity C of the condenser. With electricity fed from a constant-potential source, the current is then determined by the potential difference of

the source, and the resistance R, interposed between the condenser and the feeding points having the fixed potential difference. In short the current is $i = \frac{E}{R}$. After an infinite time, which may be practically a second or two, i will become zero, and the final charge in the condenser will be $C\,E$, a quantity independent of the resistance

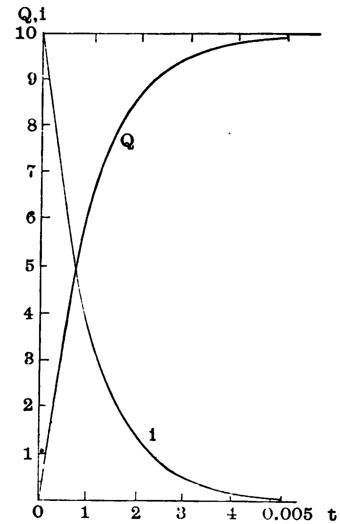


Fig. 97.

Quantity and current during the charging of a condenser, when $L = O$.

through which the electricity has been conveyed to the condenser.

Fig. 97 shows how the current varies under the conditions of charging represented in the curve Q for the same figure. The current starts at 10 amperes. The currents for various values of t are for this case computed from the equation

$$log_{10}\, i = 1 - 434.3\, t.$$

In (284) if

$$t = RC \text{.................... ... .} (285)$$

then

$$i = \frac{1}{\varepsilon}\,\frac{E}{R}.$$

Hence

$$\tan\phi' = \tan 2\pi\frac{\tau}{T} = \frac{2\pi RC}{T} \text{..........(321).}$$

This determines the angle of lag of the phase for Q on the similar phases in E. The maximum E takes place when the arc $2\pi\frac{t}{T}$ is $\frac{\pi}{2}$ or when $t = \frac{1}{4}T$. The maximum in Q is later by an arc $2\pi\frac{\tau}{T}$, where τ is the time interval of lag. Substituting the value of $\frac{2\pi RC}{T}$ of (321) in the numerator of (320) and reducing

$$Q = \frac{E_0 T^2}{4\pi^2 C \cos 2\pi\frac{\tau}{T}} \quad \frac{\sin\left(2\pi\frac{t}{T} - 2\pi\frac{\tau}{T}\right)}{R^2 + \frac{T^2}{4\pi^2C^2}} \text{...(322).}$$

Equation (321) may be written

$$\frac{\sqrt{1 - \cos^2 2\pi\frac{\tau}{T}}}{\cos 2\pi\frac{\tau}{T}} = \frac{2\pi RC}{T}, \text{ from which}$$

$$\cos\phi' = \cos 2\pi\frac{\tau}{T} = \frac{T}{2\pi C\sqrt{R^2 + \frac{T^2}{4\pi^2C^2}}} \text{.........(323).}$$

Inserting this value in (322) we have finally

$$Q = E_0\frac{T}{2\pi} \quad \frac{\sin\left(2\pi\frac{t}{T} - \phi'\right)}{\sqrt{R^2 + \frac{T^2}{4\pi^2C^2}}} \text{............. (324),}$$

where ϕ' is determined in (321). Instead of solving *(316) in* the same manner as we have solved (315) we

may determine i by simply differentiating (324). Since $\frac{dQ}{dt} = i$, we have

$$i = E_0 \frac{\cos\left(2\pi\frac{t}{T} - \phi'\right)}{\sqrt{R^2 + \frac{T^2}{4\pi^2 C^2}}}.$$

This equation may be written

$$i = E_0 \frac{\sin\left(2\pi\frac{t}{T} + \frac{\pi}{2} - \phi'\right)}{\sqrt{R^2 + \frac{T^2}{4\pi^2 C^2}}}.$$

The phase difference between E and i is, therefore, $\frac{\pi}{2} - \phi'$.

Since

$$\tan\left(\frac{\pi}{2} - \phi'\right) = \cot\phi' = \frac{T}{2\pi R C}$$

if we write $\phi = \frac{\pi}{2} - \phi'$ we may write as the equation for current

$$i = \frac{E_0}{\sqrt{R^2 + \frac{T^2}{4\pi^2 C^2}}} \sin\left(2\pi\frac{t}{T} + \phi\right) \ldots\ldots (325).$$

where

$$\tan\phi = \frac{T}{2\pi R C} \ldots\ldots\ldots\ldots\ldots\ldots (326).$$

These two equations which determine the current in a circuit having resistance and capacity, should be compared with the equations for current in a circuit having resistance and self-induction (302) and (305), which are here reproduced.

$$i = \frac{E_0}{\sqrt{R^2 + \frac{4\pi^2 L^2}{T^2}}} \sin\left(2\pi\frac{t}{T} - \phi\right) \ldots (305)$$

where

$$\tan\phi = \frac{2\pi L}{RT} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (302).$$

Summary of Results.

An inspection of these equations reveals the following results: —

In a circuit having an *E. M. F.* represented by (298), in circuits having

Resistance and Self-Induction.	Resistance and Capacity.
The current phase lags behind the phase of the impressed *E. M. F.*	The current phase is advanced on the phase of the impressed *E. M. F.*
The angle of lag is increased by increasing the resistance.	The angle of advance is increased by increasing the resistance.
Increasing the speed of the alternator which diminishes T, increases the angle of lag.	Increasing the speed of the alternator decreases the angle of advance.
Increasing the coefficient of self-induction, L, increases the angle of lag.	Increasing the capacity decreases the angle of advance.
If $L = \infty$ the angle of lag $= 90°$ but $i = 0$.	If $C = \infty$ the angle of advance $= 0$ and $i = \frac{E}{R}$
If $L = 0$ the angle of lag $= 0$ and $i = \frac{E}{R}$.	If $C = 0$, the angle of advance $= 90°$ but $i = 0$.
If $R = 0$ the angle of lag $= 90°$.	If $R = 0$ the angle of advance $= 90°$.

The angle of lag in one case is equal to the angle of advance in the other if

$$\frac{2\pi L}{R T} = \frac{T}{2\pi R C}$$

or

$$L C = \frac{T^2}{4\pi^2} \quad \text{..........................(327).}$$

154. In section 147, we found that a self-induction coefficient of $L = 0.019$ henry in a circuit of 4 ohms resistance, and with a period of 0.008 second, would produce a lag of 75°. By either (327) or (326) we may

representing energy lost,
ted from (289) as follows:

Hence $\frac{E^2}{R} = 20{,}000.$

$RC = 0.001.$

$\epsilon^{-2000\,t}$

030 — 2000 Mt

mon logarithms = 0.43429.

030 — 868.58 t.

= 3.43245

707 watts.

ints of the curve can be com-
hen $t = 0, \frac{dH}{dt} = 20{,}000.$

discharge of a condenser.

harging battery out of circuit,
ircuit through a total external
pose the condition $E = 0$ in

$$+ \frac{Q}{C} = 0 \quad\text{..................} (293).$$

ating we have

$$\int_{Q'}^{Q} \frac{dQ}{Q} = -RC \log \frac{Q}{Q'}$$

$$Q' \epsilon^{-\frac{t}{RC}} \quad\text{..................} (294).$$

We may now determine the current that would flow in such a circuit. By the conditions of the problem we have from (326)

$$\frac{T}{2\,\pi\,C} = \tan\phi\, R = 4 \times 3.73 = 14.92$$

$$\therefore \frac{T^2}{4\,\pi^2\,C^2} = 223.$$

With a maximum *E. M. F.* of $E_0 = 1200$ volts, as was assumed in section 147, we have by (325) [see (309)]

$$i = 1200 \frac{\sin(a + 75)}{\sqrt{16 + 223}}.$$

$$= 77.7 \sin(a + 75)$$

The maximum current is, therefore, exactly the same as when a circuit with no condenser and a self-induction coefficient 0.019 henry was used, or 77.7 amperes. The values of E of Fig. 100 and of i computed from the last equation, are shown in Fig. 102.

The general value for the maximum current is by (325)

$$I = \frac{E_0}{\sqrt{R^2 + \frac{T^2}{4\pi^2 C^2}}} \qquad \text{(328).}$$

By (326)

$$\frac{\sqrt{1 - \cos^2\phi}}{\cos\phi} = \frac{T}{2\pi\, R\, C}$$

$$\therefore \cos\phi = \frac{R}{\sqrt{R^2 + \frac{T^2}{4\pi^2 C^2}}} \qquad \text{(329).}$$

Substituting the radical in (328) and that equation be written

$$I = \frac{E_0}{R} \cos\phi$$

h is identical with (307).

155. *Average Power in a circuit having a condenser but no self-induction, where E is a sine function.*

By (325) and (298) the power applied at any instant is

$$\frac{dw}{dt} = Ei = \frac{E_0^2}{\sqrt{R^2 + \frac{T^2}{4\pi C^2}}} \times$$

$$\sin 2\pi \frac{t}{T} \sin\left(2\pi \frac{t}{T} + \phi\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (330).$$

By a method identical with that used in section 148, the average value of the power applied is

$$\text{Average } Ei = \frac{E_0^2}{\sqrt{R^2 + \frac{T^2}{4\pi^2 C^2}}} \quad \frac{\cos\phi}{2} \ldots\ldots\ldots (331).$$

$$= E_0\, I \frac{\cos\phi}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (332).$$

$$\text{where } I = \frac{E_0}{\sqrt{R^2 + \frac{T^2}{4\pi^2 C^2}}} \text{ and } \cos\phi = \frac{R}{\sqrt{R^2 + \frac{T^2}{4\pi^2 C^2}}}$$

(Compare (311) and (304))

The power lost by heating the resistance R is by (325),

$$\frac{dH}{dt} = Ri^2 = \frac{E^2{}_0\, R}{R^2 + \frac{T^2}{4\pi^2 C^2}} \quad \sin^2(a + \phi) \ldots\ldots\ldots\ldots (333).$$

$$\text{where } a = 2\pi \frac{t}{T}$$

As in section 131, the average $\sin^2$ is $\frac{1}{2}$. By (329) this [illegible]tion there[illegible]reduces to

$$\frac{E^2{}_0}{\sqrt{R^2 + \frac{T^2}{4\pi^2 C^2}}} \frac{\cos\phi}{2}$$

$$I \cos\phi$$

which is the same as (332). For the average power put into the condenser we have from the energy equation (281)

$$\frac{dw'}{dt} = \frac{Q}{C}\frac{dQ}{dt} = \frac{Q}{C}\,i$$

hence by (324) and (325) remembering that ϕ' in (324) is equal to $\frac{\pi}{2} - \phi$ in (325)

$$\text{and } sin\left(2\ \pi\ \frac{t}{T} + \phi - \frac{\pi}{2}\right) = - cos\left(2\ \pi\ \frac{t}{T} + \phi\right)$$

$$\frac{dw'}{dt} = i\ \frac{Q}{C} =$$

$$-\frac{T}{2\,\pi\,C}\ \frac{E_0^2}{R^2 + \frac{T^2}{4\,\pi^2\,C^2}}\ sin\ (a + \phi)\ cos\ (a + \phi)$$

$$\text{where } a = 2\,\pi\frac{t}{T}.$$

This result should be compared with (315). Here as in that case from the consideration of section 148, the average $sin \times cos$ between zero and 180° is zero. It, therefore, follows that on the average, $\frac{dw'}{dt} = 0$. The condenser consumes no power, but continually receives and restores it in a periodic manner, which we shall proceed to investigate.

The average power represented in (331) with an angle of advance of 75° is the same as that represented in (311) with an angle of lag of 75°, if the values of E_0 and of I are made equal in the two cases. The capacity equivalent to a given self-induction yielding the same average power at a given speed is, therefore, given in (327).

156. *Power at any instant in a circuit having resistance and capacity where E is periodic.*

From equations (298) and (325) we have for the impressed power at any instant

$$\frac{dw}{} = Ei = E_0\,I\,sin\,a\,sin\,(a + \phi) \quad \ldots\ldots\ldots\ldots (334).$$

Assuming the data heretofore used in discussing self-induction, with an angle of 75°, we have

$$\frac{dw}{dt} = 1200 \times 77.7 \sin a \sin (a + 75)$$
$$= 93240 \sin a \sin (a + 75).$$

(See section 151.) The capacity yielding this result is by section 154, 8.53×10^{-6} farads. The value of Ei for all values of a is shown in Fig. 103.

$$\text{If } a = 10°, \text{ then } a + 75 = 85°$$

$$\frac{dw}{dt} = Ei = 93240 \times 0.1736 \times 0.9962$$
$$= 16120 \text{ watts.}$$

The power consumed in heating the circuit at any instant is (333)

$$\frac{dH}{dt} = Ri^2 = RI^2 \sin^2 (a + \phi) \ldots\ldots\ldots (335).$$
$$= 4 \times (77.7)^2 \sin^2 (a + 75)$$
$$= 24130 \sin^2 (a + 75) \text{ watts.}$$

For $a = 10°, \quad a + 75 = 85°$

$$\text{or } Ri^2 = 23940 \text{ watts.}$$

The value of Ri^2 for all values of a is represented in Fig. 103.

The power stored in the condenser at any instant is given in (334).

We have

$$\frac{dw'}{dt} = i\frac{Q}{C} = -\frac{T}{2\pi C} I^2 \sin (a + \phi) \cos (a + \phi) \ldots (336).$$

By section 154

$$\frac{dw'}{dt} = \frac{Q}{C} i = -14.92 \times (77.7)^2 \sin (a + \phi) \cos (a + \phi)$$
$$= -90,000 \sin (a + 75) \cos (a + 75)$$

If $a = 10° \qquad a + 75 = 85°$

$$\frac{dw'}{dt} = -7820 \text{ watts.}$$

When $a = 10^{\circ}$, therefore the system is receiving 16120 watts from the supply, the condenser is discharging 7820 watts, and all of this power, or 23940 watts is being consumed in heating the resistance, R.

157. *Quantity of electricity in the condenser during a cycle.*

The quantity in the condenser at any instant during a cycle is given by Eq. (324). In order that this equation may be compared with those of $E. M. F.$ and current, we must place for ϕ' its value $\frac{\pi}{2} - \phi$. The equation then becomes

$$Q = - E_0 \frac{T}{2\pi} \frac{\cos(a + \phi)}{\sqrt{R^2 + \frac{T^2}{4\pi^2 C^2}}} \qquad (337).$$

$$= - \frac{T}{2\pi} I \cos(a + \phi) \qquad (338).$$

where

$$\tan \phi = \frac{T}{2\pi RC}$$

Substituting in (338) the values corresponding to an angle ϕ of 75° (see sections 154, 147)

$$Q = - \frac{0.008}{2\pi} 77.7 \cos(a + 75)$$

$$= - 0.0989 \cos(a + 75) \qquad (339).$$

The maximum charge will occur when $\cos(a + 75) = \pm 1$ or when $a = - \phi$ or $180 - \phi$. In the particular case being discussed the value of a will be -75 or $180 - 75$. The charge in the condenser will then be $\pm$ 0.0989 culombs.

The maximum potential of the condenser will be

$$V = \frac{Q}{C} = \pm \frac{E_0 T}{2\pi C} \frac{1}{\sqrt{R^2 + \frac{T^2}{4\pi^2 C^2}}} \qquad (340).$$

By (323) we have

$$\cos \phi' = \frac{T}{2\pi C \sqrt{R^2 + \frac{T^2}{4\pi^2 C^2}}}$$

Since $\phi' = \frac{\pi}{2} - \phi$ we have

$$\cos \phi' = \sin \phi = \frac{T}{2\pi C \sqrt{R^2 + \frac{T^2}{4\pi^2 C^2}}} \text{......} (341).$$

Hence (340) becomes

$$V = \pm E_0 \sin \phi \text{.....................} (342).$$

If the maximum potential of the supply mains were 1200 therefore, the maximum potential of the condenser when the angle of advance ϕ is 75° would be 1200 $\times$ 0.9659, or 1159 volts. Equation (340) shows that if the period T were very small, as in the alternations which constitute light, the condenser potential would be unaffected by connection with such supply mains. The frequency of the lowest red which is, visible, is, according to Langley, 3.92×10^{14} per second. If, therefore, in (340) we make $R = 4$ ohms, $E_0 = 1200$ volts, $C = 8.53 \times 10^{-5}$ farad as in the particular case discussed at length in the preceding sections, the maximum potential V of the condenser becomes for this value of $T = \frac{1}{3.92 \times 10^{14}}$.

$$V = \frac{1200}{4\pi \times 8.53 \times 10^{-5} \times 3.92 \times 10^{14}}$$
$$= 0.0000000028 \text{ volts.}$$

The equation also shows that if the capacity C be made correspondingly small, it will counteract the effect of making T small. The pupil should investigate the effect of such enormous frequencies on the angles of lag and advance, and the possibilities of transmitting power electrically with such frequencies. The pupil should also construct the curve for Q from (339) and plot the curves

for i and E, as they are given in Fig. 102, upon the same sheet.

If R were zero in a condenser circuit the angle of advance would be 90° and the values of Q, i, maximum V, from (337) (325) and (342) become

$$Q = -E_0 C \cos a$$

$$i = \frac{2\pi C}{T} E_0 \sin a$$

$$V_{max} = \pm E_0$$

158. *Graphical Representation of a periodic E. M. F. following the law of sines.*

Well designed alternators usually yield an $E. M. F.$ which follows the law of sines, given in Eq. (298), or

$$E = E_0 \sin 2\pi \frac{t}{T} = E_0 \sin \omega t.$$

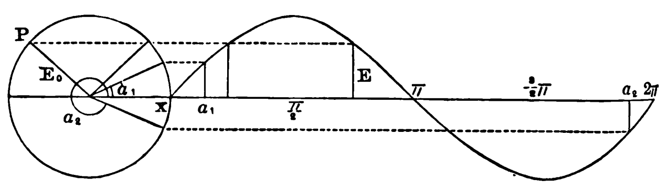


Fig. 104.
Sine Curve.

The variation in E during a cycle may be represented by means of Fig. 104. Draw a circle having a radius E_0. Suppose a point P to travel around the circle with a constant angular velocity $\frac{2\pi}{T}$. The vector E_0, will sweep over the angle 2π in a time T. Estimating angles from the horizontal radius at x, the angle swept over in a time t will be $\frac{2\pi}{T} t = \omega t$. The length of the projection of E_0 on the vertical diameter is

$$E = E_0 \sin \omega t.$$

Let $x\ 2\pi$ be the rectified circumference $2\pi E_0$ along which we may suppose a particle P' to travel with the same linear

velocity that P has on the arc of the circle. Let the two points start out from x at the same instant. While P is travelling around the circle to x again, P' will travel uniformly over the equal distance $x\ 2\pi$, when we may suppose it to return instantaneously to x in order to start with P again. For each position of P' erect an ordinate equal in length to the projection of E_0 on a vertical radius. The summits of these ordinates will lie in the curve of sines, if E_0 is unity. When E_0 is greater than unity, the circle and curve are drawn on a larger scale, the linear distances being multiplied by E_0.

The area of the curve during half a period or from 0 to $2\ \pi$ in arc is easily obtained. The element of the area $d\ A$ is

$$E\ dx = E \times E_0\ \omega\ dt \text{ or}$$
$$dA = E_0^2\ sin\ \omega t \times \omega\ dt$$

Integrating between 0 and π

$$A = E_0^2 \int_0^\pi sin\ \omega t \times \omega dt = E_0^2 \Big[-cos\ \omega t \Big]_0^\pi$$
$$= -E_0^2 \Big(-1 - 1 \Big) = 2\ E_0^2.$$

The average E will, therefore, be the average ordinate of the curve and will be

$$\text{Average } E = \frac{2\ E_0^2}{\pi\ E_0} = \frac{2}{\pi} E_0 \text{ ....(343).}$$

When $E_0 = 1$ the average ordinate is $\frac{2}{\pi}$ which is, therefore, the average sine. The average $E.\ M.\ F.$ when E varies according to the law of sines is, therefore, $\frac{2}{\pi} E_0$ if E_0 is the maximum value.

If the ordinates of this curve were all squared, and the squares laid off as ordinates of a new curve, the element of area would be $E^2 \times E_0\ \omega dt.$

$$dA' = E_0^3\ sin^2\ \omega t \times \omega dt.$$

Integrating between zero and π we have the area of this curve, and by dividing by the corresponding arc we have the mean value of E^2. We thus have

$$A = E^2_0 \int_0^{\frac{T}{2}} sin^2\ \omega t \times \omega dt$$

$$= E^2_0 \left[\frac{\omega t}{2} - \frac{1}{4}\ sin\ 2\ \omega t \right]_0^{\frac{T}{2}}$$

$$= \frac{\pi\ E_0^2}{2}$$

Dividing by $\pi\ E_0$ we have for the mean of the squared values of E

$$E_v = \frac{\pi\ E_0^2}{2\pi\ E_0} = \frac{E_0^2}{2}$$

$$\therefore\ E_v = \frac{E_0}{\sqrt{2}}$$

$$\text{or } E_0 = \sqrt{2}\ E_v \text{.................................} (344).$$

As in section 133, a hot wire voltmeter standardized for continuous currents gives directly the reading E_v with alternating currents, since the heating effects depend on the average values of Ri^2. Reading the voltage on such an instrument we obtain E_0 at once by multiplying the reading by $\sqrt{2} = 1.41$. In the special case discussed at length in the preceding sections, where E_0 was taken as 1200 volts, the Cardew Voltmeter reading would be $\frac{1200}{\sqrt{2}} = 851$ volts. The voltmeter reading is called the virtual potential. It differs materially from the average potential (343) which is 763 when E_0 is *1200.*

159. *Graphical representation of a periodic current in a circuit having resistance and self-induction.*

The preceding discussion has given for the current in a circuit having resistance and self-induction,

$$i = I \sin(\omega t - \phi)$$

$$I = \frac{E_0}{\sqrt{R^2 + \frac{4\pi^2 L^2}{T^2}}}$$

$$\tan\phi = \frac{2\pi L}{R\,T}.$$

It is evident that in the equation for I the quantities are related as the sides and hypotenuse of a right-angled triangle are related. We may write

$$E^2_0 = R^2 I^2 + \omega^2 L^2 I^2$$

where $\omega = \frac{2\pi}{T}$ which is the angular velocity. If ϕ be the angle between E_0 and RI in this triangle, then the tangent of this angle is

$$\tan\phi = \frac{\omega L}{R} = \frac{2\pi L}{R\,T}$$

which is the angle of lag of current on electro-motive force.

It is, therefore, evident that if we prolong the line R I to I in Fig 105 the entire length of the line as prolonged will represent the maximum current if the length of the prolongation is such that the whole length is to the side R I as I is to $R\,I$. The construction shown in Fig. 105 assumes that R is less than unity. If it be greater, the vector I would be less than that representing $R\,I$.

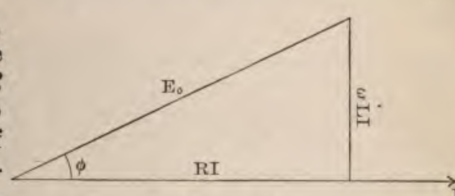


Fig. 105.
Triangle of electro-motive force.

We have shown that i is a periodic sine function, and this is represented in the first equation at the head of this section. It is, therefore, evident that if the whole tri-

angle Fig. 105 be revolved about the point at the vertex of the angle ϕ, as the vector E_0 is revolved in Fig. 104, the projection of I on the vertical line Fig. 106 will give the current at any time.

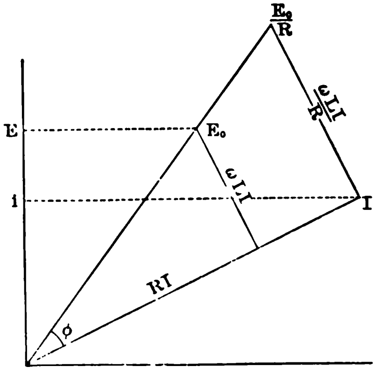


Fig. 106.
Triangles of electromotive force and current.

If we complete a similar right-angled triangle having the side I as a base, its hypotenuse will evidently be the hypotenuse of the former triangle divided by R, or $\frac{E_0}{R}$, and its remaining side will be $\frac{\omega L I}{R}$. (See Fig. 106.) If ϕ become smaller and finally become zero the sides $\omega\, L\, I$ and $\frac{\omega\, L\, I}{R}$ become zero. This may be because $L = 0$. Then the side $R\ I$ will equal the hypotenuse E_0 and the side I will equal the hypotenuse $\frac{E_0}{R}$, which conditions satisfy Ohm's law.

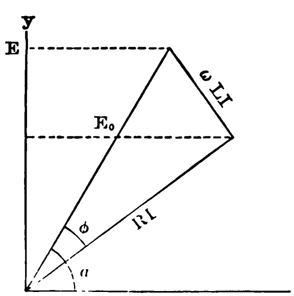


Fig. 107.
Resolved components of $E.\,M.\,F.$, E_0.

Referring to the smaller triangle which is reproduced in Fig. 107, it is evident that the sides $R\ I$ and $\omega\ L\ I$ are components of E_0. The sum of their projections on the axis y is equal to E.

From the energy equation we have as the value of E,

$$E = R\,i + L\,\frac{di}{dt}$$

or putting in the values of E, i and $\frac{di}{dt}$, since $i = I \sin(\omega\, t - \phi)$

$$E = E_0 \sin \omega t = R\, I \sin(\omega t - \phi) + L\omega I \cos(\omega t - \phi) \quad (345)$$
$$= R I \sin(\omega t - \phi) + L\omega I \sin(\omega t - \phi + 90°).$$

In Fig. 107 the angle α represents $\omega\, t$, and it is evident that the two terms of the last equation are the projections of the sides $R\, I$ and $L\, \omega\, I$ on the axis y. The component $\omega\, L\, I$ is shown by the last equation to be advanced in phase 90° ahead of the component $R\, I$ and this is also shown in the figure. The effective $E.\, M.\, F.$

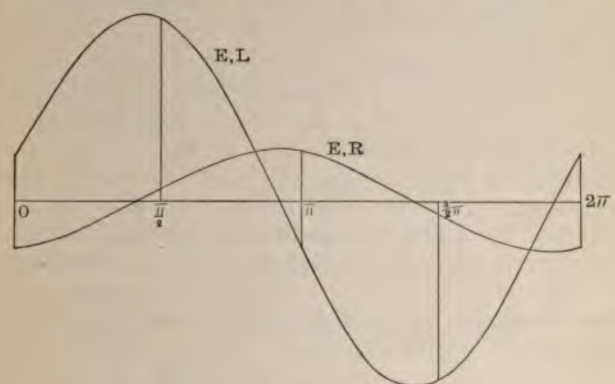


Fig. 108.

Components of $E.\, M.\, F.$ which balance self-induction and maintain the current through the resistance.

due to one component will be zero when that due to the other is at its maximum.

The component $\omega\, L\, I$ is that part of the impressed $E.\, M.\, F.$ which balances the back $E.\, M.\, F.$ due to the self-induction of the coil, and the component RI is the part which maintains the current through the ohmic resistance. The $E.\, M.\, F.$ which maintains the current is represented by a vector whose direction is the same as that of the vector representing I. The $E.\, M.\, F.$ which balances the self-induction of the coil is represented by a vector at right angles to the former and is equal and opposite to the vector which represents the back $E.\, M.\, F.$ due to the self-induction of the coil.

We shall apply Eq. (345) to the conditions represented in Figures 100 *and* 101, in which $R = 4$ ohms, $L =$ [illegible]

currents would be equal if $R_2 = \omega L_1$, a relation which would, therefore, hold for a definite value of ω, or for a definite speed of the alternator. No heat would be generated and lost in the circuit having only self-induction, no iron core being present, but energy stored in this circuit would be wholly reclaimed, during each period.

The total maximum current supplied to the two branches would be (350)

$$I = \frac{E_0}{R_2 \omega L_1} \sqrt{R_2^2 + \omega^2 L_1^2}$$

which when $R_2 = \omega L_1$ becomes

$$I = \frac{E_0}{R_2} \sqrt{2} = \frac{E_0}{\omega L_1} \sqrt{2}.$$

The angle of lag in the equivalent circuit would be determined by the equation (352)

$$\cos \phi = \frac{\omega L_1}{\sqrt{R_2^2 + \omega_2 L_1^2}}$$

which when $R_2 = \omega L_1$ becomes

$$\cos \phi = \frac{1}{\sqrt{2}} \text{ or } \phi = 45°.$$

The resistance of the equivalent circuit becomes (353)

$$R = \frac{\omega^2 L_1^2}{R_2^2 + \omega^2 L_1^2} R_2$$

When $R_2 = \omega L_1$ we have $R = \dfrac{R_2}{2}$

The coefficient of self-induction becomes (354)

$$L = \frac{R_2^2}{R_2^2 + \omega^2 L_1^2} L_1$$

which when $R_2 = \omega L_1$ becomes $L = \dfrac{L_1}{2}$. The pupil may discuss the similar case for circuits having resistance and capacity.

168. *Circuits having resistance self-induction and capacity.*

The solution of the general equation for this case is deemed too difficult for the grade of instruction for which this treatise is written, but the results of such a discussion

in the circuit, or $L = 0$, the second term of the last member of the equation becomes zero. Since $\phi = 0$

$$i = I \sin \omega t = \frac{E_0 \sin \omega t}{R}$$

$$\text{or } I = \frac{E_0}{R} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (347)a$$

or maximum current is equal to the maximum $E. M. F.$ divided by the ohmic resistance, which satisfies Ohm's law.

If the ohmic resistance in the circuit be zero, then (302) $\phi = 90°$ and

$$L \omega I \sin \omega t = E_0 \sin \omega t$$

$$\text{or } I = \frac{E_0}{L \omega} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (347)b.$$

Both equations follow at once from (306). If we impose the conditions demanded by (347)a and make $E_0 = 1200$ volts, and $R = 4$ ohms, then $I = 300$ amperes. The virtual current measured by a Thomson ampere balance, or a Siemens electro-dynamometer would then be $\frac{300}{\sqrt{2}} = 212$ amperes. If we impose the conditions demanded by Eq. (347)b and make $E_0 =$ 1200, $L = 0.019$ henry and $T = 0.008$ second or $\omega = \frac{2\pi}{T} = 785.4$ radians, then

$$I = \frac{1200}{0.019 \times 785.4} = 80.4 \text{ amperes.}$$

The virtual current shown by the amperemeter would then be 56.4 amperes. The heat generated in the circuit would be zero. The power being stored in the magnetic field during part of a cycle would be wholly recovered during the cycle. The positive and negative loops in the curve Ei of Fig. 101 would be equal to each other, and the curve $Li \frac{di}{dt}$ would be identical with it.

160. *Effect of varying resistance.*

The effect of varying resistance may be determined by means of triangles of electromotive force and current of

Fig. 106, which are reproduced in Fig. 109. The vector $O\,E_0$ representing the maximum impressed $E.\ M.\ F.$ is placed vertically. The current vector makes an angle ϕ with vector E_0. The current vectors $\frac{E_0}{R}$ and I are in the same directions respectively as E_0 and $R\,I$, and their length is determined by dividing the latter quantities by R. As we have seen, $\frac{E_0}{R}$ is the value that I would have if ϕ were zero. If ϕ were to be increased while R were constant, the values E_0 and $\frac{E_0}{R}$ being constant, it is evident that $R\,I$ and I would be represented by chords of two circles having diameters E_0 and $\frac{E_0}{R}$ these chords being drawn through O and making an angle ϕ with vector E. In the $E.\ M.\ F.$ circle the lower chord $R\,I$ represents the maximum of that part of the impressed $E.\ M.\ F.$ concerned in heating the wire, and $\omega\,L\,I$ is the part which opposes the $E.\ M.\ F.$ of self-induction. As ϕ increases, the upper chord lengthens and the lower one shortens. ϕ may be increased by increasing either ω or L, or by diminishing R, as is shown by the equation for ϕ, viz. (302)

$$tan\,\phi = \frac{\omega L}{R}.$$

Multiplying the last equation by E_0 it may be put in the form

$$\frac{E_0}{R} = \frac{E_0}{\omega L}\,tan\,\phi$$

By the last section $\frac{E_0}{\omega L}$ is the value of I when $R = 0$ and $\phi = 90°$. Therefore, if we draw a horizontal line through O, making the length represent $\frac{E_0}{L\omega}$, and draw the hypotenuse to the upper end of line $\frac{E_0}{R}$, this line will *pass through* the end of the line representing I.

If we now decrease R keeping L constant, $\frac{E_0}{R}$ will increase, ϕ will increase, but the side $\frac{E_0}{L\omega}$ will be unchanged. Therefore I will always be represented by a chord of the semi-circle having a diameter $\frac{E_0}{L\omega}$ and when $R = 0$, ϕ will equal 90° and I will equal $\frac{E_0}{L\omega}$. If R increases indefinitely, I will become zero. The variation of I with

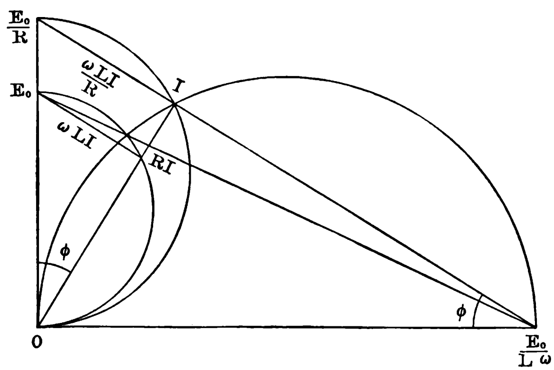


Fig. 109.
Limiting cases of inductive resistances.

values of R from 0 to 0.8 in a special case, is shown in Fig. 110, which is drawn to scale, where $E_0 = 6$ volts, $L\omega = 0.6$, and, therefore, $\frac{E_0}{L\omega} = 10$. Values of R greater than unity are not represented in order to avoid complicating the figure. It is evident from (305) that when $R = 0$, $i = -\frac{E_0}{L\omega} \cos \omega t$. The entire $E.\ M.\ F.$ of the source will be balanced against the back $E.\ M.\ F.$ of self-induction. In Fig. 101 the area of the negative loops of the power curve Ei will then be equal in area to that of the positive loops.

The *conditions* represented in Fig. 110 are reproduced

in Fig. 111, where, however, the resistance R and, therefore, $\frac{E_0}{R}$ is kept constant at one of the values represented in Fig. 110, and the value of $L\omega$ is varied from 0 to unity. Since I is a common chord in two circles, one of which is determined by $L\omega$ and the other by R, it is evident that a similar variation of I will exist here as in the other case. The change in $L\omega$ may be due either to

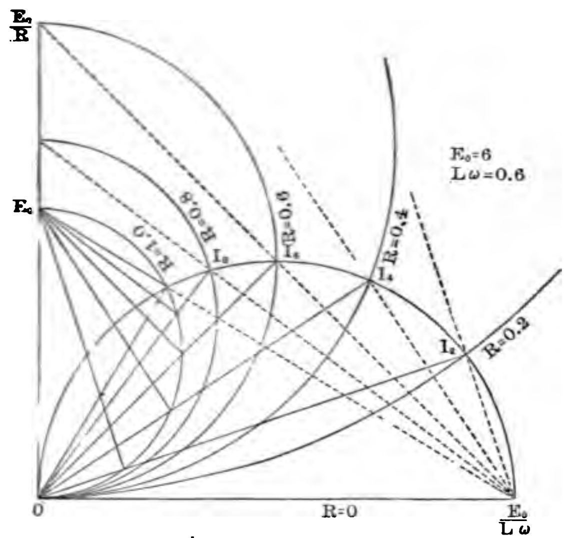


Fig. 110.

Current through inductive resistance. Resistance varied.

a change in L or in ω. If the speed of an alternator be uniformily increased, any transformer connected to its mains would be traversed by currents represented by the diminishing chords of the outer circle in Fig. 111. If the value of $L\omega$ could be made infinite ϕ would become 90° and the current would be zero.

Increasing the resistance, diminishes the angle of lag, and diminishes the current. Increasing the self-*induction* co-efficient L increases the angle of lag and *diminishes* the current. Increasing the angular velocity

$\omega = \frac{2\pi}{T}$, increases the angle of lag and diminishes the current. These relations are shown in equations (302) and (306) and also by Figs. 110 and 111.

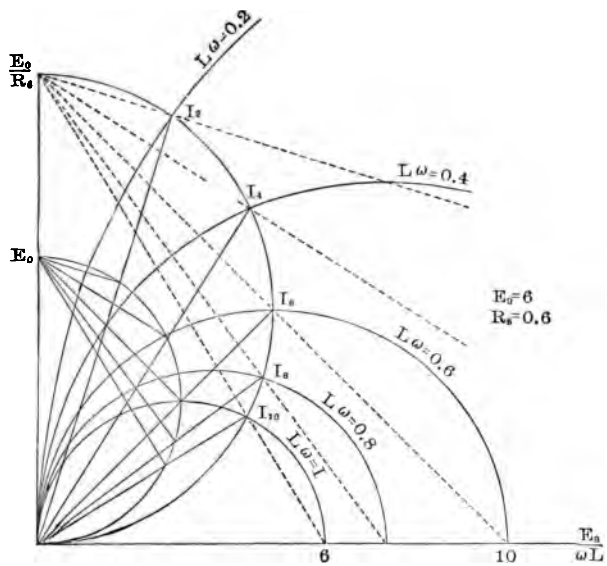


Fig. 111.
Current through inductive resistance. Inductance varied.

161. *Inductive resistances in series.*

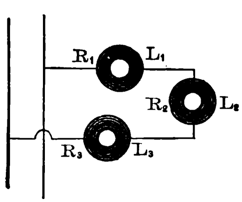


Fig. 112.
Inductive resistances in series.

Let E_0 be the maximum potential between two mains upon which we will assume current and $E.\ M.\ F$ to be in unison. A circuit connected with these mains has inductive resistances $L_1\ R_1$, $L_2\ R_2$, etc., in series. Let it be required to find E_0 in order that a given current may be forced through these resistances, and we are to determine the angle of lag due to the system. Draw $O\ I$ to represent the given

current I. (Fig. 113.) Lay off $R_1 I$ from O representing the $E.\ M.\ F.$ required to maintain the current I through this resistance if there were no self-induction, and draw a perpendicular at the end of this line whose length shall represent $L_1\, \omega I$. This will represent the $E.\ M.\ F.$ required to balance the self-induction of this coil. The hypotenuse of the triangle of which these two lines are the sides will represent the impressed E'_0

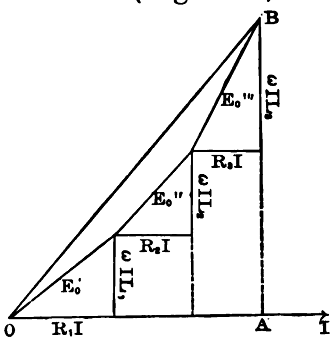


Fig. 113.
Inductive resistances in series.

required to maintain the current through this part of the resistance. This vector E'_0 must be in advance of I by an angle determined by the condition $tan\ \phi' = \frac{2\pi L_1}{R_1 T} = \frac{\omega L_1}{R_1}$. From the upper end of the perpendicular lay off $R_2 I$ parallel to $O\ I$, and at its end erect the perpendicular $L_2\ \omega I$. This will determine the triangle of $E.M.F.$'s for the second coil. Continue in this manner until the chain of triangles terminating in B includes all the resistances of the circuit. The line $O\ B$ will then represent the $E.\ M.\ F.$ that must be impressed on the system. The angle of lag of the current I on this impressed E_0 will be $B\ O\ A = \phi$, determined by

$$tan\ \phi = \frac{\omega \Sigma L}{\Sigma R}.$$

The single coil which would replace the system and yield the same results, must have a self-induction coefficient $L = \Sigma\ L$ and a resistance $\Sigma\ R$. In this coil the $E.\ M.\ F.$ maintaining the current I will be $\overline{O\ A} = I\, \Sigma\ R$ and the $E.\ M.\ F.$ required to balance self-induction will be $\overline{A\ B} = I\, \omega\, \Sigma\ L$. Any one of these resistances having no self-induction, the triangle of $E.\ M.\ F.$ for this resistance would have its hypotenuse coincident with its base. If it had no resistance its base would be zero.

Suppose $R_1 = 0.1, R_2 = 0.2, R_3 = 0.4$ ohms and the current maximum $= 20$ amperes. Then $R_1 I = 2$; $R_2 I = 4$ and $R_3 I = 8$. Then the resistance of the equivalent coil would be 0.7 ohm, and the *E. M. F.* maintaining the current through the resistances would be $I \Sigma R = 14$ volts. The power delivered to the system as heat per second would be $\frac{1}{2} I^2 \Sigma R = 140$ watts. The *E. M. F.* required to balance the self-induction *E. M. F.* would be $\omega I \Sigma L$. If $\omega = \frac{2\pi}{T} = 800$, $L_1 = 0.002$, $L_2 = 0.001$, $L_3 = 0.004$ henrys, then this *E. M. F.* is $800 \times 20 \times 0.007 = 112$ volts. The angle of lag of current on impressed *E. M. F.* E_0 would be determined by

$$\tan \phi = \frac{800 \times 0.007}{0.7} = 8.0$$

$$\therefore \phi = 82°53'.$$

The maximum impressed *E. M. F.* would be

$$E_0 = \frac{112}{\sin \phi} = \frac{112}{0.9923} = 112.8 \text{ volts.}$$

The virtual current which determines the heating effect and which would be read directly on a Siemens electro-dynamometer standardized for continuous currents would be (section 158) $\frac{20}{\sqrt{2}}$ amperes. The virtual impressed *E. M. F.* read on a hot wire voltmeter similarly graduated would be $\frac{112.8}{\sqrt{2}}$ volts. (section 158). The power delivered to the system would, therefore, be (Equations 331–2)

$$\text{Average } Ei = \frac{112.8}{\sqrt{2}} \times \frac{20}{\sqrt{2}} \cos \phi$$

$$= \frac{1}{2} E_0 I \cos \phi = \frac{1}{2} \times 112.8 \times 20 \times 0.1239, = 140 \text{ watts,}$$

as was before obtained.

If the impressed E_0 be given, with ω and the various values of R and L, and it is desired to compute the resulting current, any current may be assumed as the resulting current, and a value of E_0 which would yield that current may be computed as just explained. This chain of triangles will be similar to the one desired, and the latter can, therefore, be computed from the former by simple proportion.

The manner of computing the power consumed by each of the resistances of the system is evident from the explanations given.

162. *Inductive Resistances in multiple.*

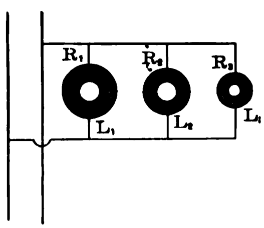


Fig. 114.
Inductive resistances in multiple.

If a series of inductive resistances R_1, L_1; R_2, L_2; etc., be connected in multiple between mains having a given maximum voltage E_0, we are required to find the current in each, and to find the single inductive resistance which will replace the system and give the same current and lag. First let two resistances R_1L_1; and R_2L_2 be discussed. Draw a semi-circle on E_0 as a diameter. The angles of lag ϕ_1 and ϕ_2 being known from the equation

$$tan\ \phi_1 = \frac{\omega L_1}{R_1} \text{ etc.}$$

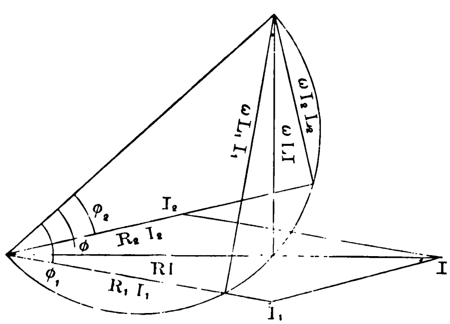


Fig. 115.
Inductive resistances as in multiple.

the chords $R_1 I_1$ and $R_2\ I_2$ are known, since $R_1\ I_1 = E_0$ $cos\ \phi_1$, and $R_2\ I_2 = E_0\ cos\ \phi_2$. These chords represent the $E.\ M.\ F.s$ which would be required to maintain the currents I_1 and I_2 in the two resistances R_1 and R_2 if there were no self-induction. Dividing

these $E.\ M.\ F.$'s by R_1 and R_2 we have (compare Eq. (307)),

$$I_1 = \frac{E_0}{R_1} \cos \phi_1 \qquad I_2 = \frac{E_0}{R_2} \cos \phi_2 \text{..........} (348).$$

As has been already shown, these currents are to be laid off along the vectors R_1 I_1 and R_2 I_2, since these currents lag behind E in phase by the angles ϕ_1 and ϕ_2.

In the equivalent inductive resistance these two component currents will be superposed exactly as the two component $E.\ M.\ F.$'s of the triangle of $E.\ M.\ F.$ are superposed in the discussion leading to equation (345). As the vectors E_0 I_2 and I_1 revolve, in unison with the periodic changes in the armature of the alternator, the vectors E_0, I_2 and I_1 revolve with an angular velocity ω. They preserve fixed relations to each other. The projections of I_2 and I_1 on the vertical axis o y, determine the currents in the two branches. I_2 will be a maximum when $\omega t - \phi_2 = 90$ and it will be zero when $\omega t - \phi_2 = 0$. In Fig. 116 the projection of I_1 on o y is

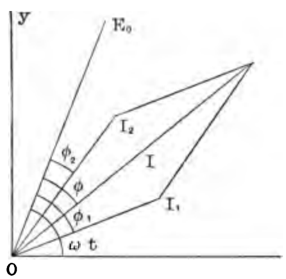


Fig. 116.
Currents as directed quantities.

$$i_1 = I_1 \sin (\omega t - \phi).$$

The projection of I_2 is

$$i_2 = I_2 \sin (\omega t - \phi_2).$$

The current in the equivalent inductive resistance will, therefore, be

$$i = i_1 + i_2 = I_1 \sin (\omega t - \phi_1) + I_2 \sin (\omega t - \phi_2).$$

The sum of these two projections is evidently equal to the projection of the diagonal of the parallelogram of which I_1 and I_2 are the sides. The angle of lag of this diagonal being indicated by ϕ, the projection of the diagonal is

$$i = I \sin (\omega t - \phi) \text{........................} (349).$$

This *equation has* the same form as (305) if the value

of I of (306) be substituted therein. It remains to determine the values of I and ϕ.

By trigonometry, from the figure

$$I^2 = I_1^2 + I_2^2 - 2I_1 \, I_2 \cos \left[180 - (\phi_1 - \phi_2)\right]$$

$$\therefore \quad I = \sqrt{I_1^2 + I_2^2 + 2I_1 \, I_2 \cos\,(\phi_1 - \phi_2)} \ldots\ldots\ldots(350)$$

which determines I, since I_1 and I_2 are known by (348) and since

$$\tan \phi_1 = \frac{\omega L_1}{R_1} \qquad \tan \phi_2 = \frac{\omega L_2}{R} \ldots\ldots\ldots\ldots(351)$$

which are given quantities.

We have also from Fig. 116,

$$\cos\,(\phi - \phi_2) = \frac{I_2^2 + I^2 - I_1^2}{2I_2 \, I}$$

in which we may place the value of I from (350). We have then

$$\cos\,(\phi - \phi_2) = \frac{I_2 + I_1 \cos\,(\phi_1 - \phi_2)}{\sqrt{I_1^2 + I_2^2 + 2I_2 I_1 \cos\,(\phi_1 - \phi_2)}} \ldots(352).$$

Knowing ϕ we may determine the resistance of the equivalent circuit. We have

$$R\,I = E_0 \cos \phi$$

$$\therefore R = \frac{E_0}{I} \cos \phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(353).$$

For the self-induction coefficient of the equivalent resistance, by trigonometry from Fig. 115,

$$\omega \, L \, I = E_0 \sin \phi$$

$$\therefore \quad L = \frac{E_0}{\omega I} \sin \phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(354).$$

Example. A hot wire voltmeter on the mains of an alternator reads 85.1 volts. Two inductive resistances connected in multiple between these mains have characterizing constants $R_1 = 0.1$ ohm, $R_2 = 0.5$ ohm, $L_1 = 0.002$ henry, $L_2 = 0.003$ henry.

The angular velocity $\omega = 800 = \frac{2\pi}{T}$, radians per second.

By (323) we have

$$\cos\phi' = \frac{T}{2\pi C\sqrt{R^2 + \dfrac{T^2}{4\pi^2 C^2}}}$$

Since $\phi' = \dfrac{\pi}{2} - \phi$ we have

$$\cos\phi' = \sin\phi = \frac{T}{2\pi C\sqrt{R^2 + \dfrac{T^2}{4\pi^2 C^2}}} \text{......} (341).$$

Hence (340) becomes

$$V = \pm E_0 \sin\phi \text{ ....} (342).$$

If the maximum potential of the supply mains were 1200 therefore, the maximum potential of the condenser when the angle of advance ϕ is 75° would be 1200 × 0.9659, or 1159 volts. Equation (340) shows that if the period T were very small, as in the alternations which constitute light, the condenser potential would be unaffected by connection with such supply mains. The frequency of the lowest red which is, visible, is, according to Langley, 3.92×10^{14} per second. If, therefore, in (340) we make $R = 4$ ohms, $E_0 = 1200$ volts, $C = 8.53 \times 10^{-5}$ farad as in the particular case discussed at length in the preceding sections, the maximum potential V of the condenser becomes for this value of $T = \dfrac{1}{3.92 \times 10^{14}}$.

$$V = \frac{1200}{4\pi \times 8.53 \times 10^{-5} \times 3.92 \times 10^{14}}$$
$$= 0.0000000028 \text{ volts.}$$

The equation also shows that if the capacity C be made correspondingly small, it will compensate the effect of making T small. The pupil should investigate the effect of such enormous frequencies on the angles of lag and advance, and the [illegible] electrically with such frequencies. [illegible] *construct the* [illegible]

By (353) the resistance of the equivalent circuit is

$$R = \frac{120}{123.29} \cos 83° 14'$$
$$= 0.1147 \text{ ohm.}$$

By (354) the coefficient of self-induction of the equivalent circuit is

$$L = \frac{120}{800 \times 123.29} \sin 83° 14'$$
$$= 0.00120 \text{ henry.}$$

Hence the current in the equivalent circuit would be at any instant

$$i = 123.29 \sin (\omega t - 83° 14')$$

where

$$I = 123.29 = \frac{120}{\sqrt{(0.1147)^2 + 800^2 (0.0012)^2}}.$$

The currents in the two branches will be

$$i_1 = 74.65 \sin (\omega t - 86° 26')$$
$$i_2 = 48.94 \sin (\omega t - 78° 14').$$

These values of i, i_1 and i_2 may be computed and plotted for every 10° from 0 to 360° with the corresponding value of E which is

$$E = 120 \sin \omega t.$$

If $\omega t = 83° 14'$ then $i = 0$. At this instant we shall have

$$i_2 = - i_1 = 4.165 \text{ amperes.}$$

At this instant the current in the main line will be zero, and the current will be flowing around the two branches, current i_1 being then negative with respect to E, a condition of flow which will be reversed 180° later.

The pupil should compute and plot the power curves for each branch and for the equivalent circuit as is done in Fig. 101.

Having determined the circuit equivalent to two parallel branches, this may be combined with a third by the same method. By the method of the previous section

it may be combined with a circuit in series with the two parallel branches.

It is evident that if in case of two circuits in multiple we have $R_1 = 0$, and $L_2 = 0$ the angles of lag will be

$$tan\ \phi_1 = \infty \text{ and } \phi_1 = 90$$
$$tan\ \phi_2 = 0 \quad \text{"} \quad \phi_2 = 0$$

The currents will differ in phase by 90°, the circuit having no self-induction being in unison with the impressed *E. M. F.* The currents will then be at any instant

$$i_1 = -I_1 \cos \omega\, t = -\frac{E_0}{\omega L_1} \cos \omega\, t$$

$$i_2 = I_2 \sin \omega t = \frac{E_0}{R_2} \sin \omega t.$$

These currents will have the same average value and the same virtual value if $R_2 = \omega L_1$.

The maximum current in the equivalent circuit would then be by (350)

$$I = \frac{E_0}{\omega R_2 L_1} \sqrt{R_2^2 + \omega^2 L_1^2}$$

and the angle of lag in the undivided circuit or in the equivalent inductive resistance would be by (352)

$$\cos \phi = \frac{\omega L_1}{\sqrt{R_2^2 + \omega^2 L_1^2}} \quad or \sin \phi = \frac{R_2}{\sqrt{R_2^2 + \omega^2 L_1^2}}$$

$$\therefore \quad \tan \phi = \frac{R_2}{\omega L_1}.$$

The condition which makes the two branch currents equal, makes $\phi = 45°$.

The current in the branch having no self-induction would be in unison with the *E. M. F.* of the undivided circuit, and 45 degrees in advance of the current in the undivided circuit. The resistance of the equivalent circuit would be by (353)

$$R = \frac{R_2\, \omega^2 L_1^2}{R_2^2 + \omega^2 L_1^2}$$

and the coefficient of self-induction would be by (354)

$$L = \frac{L_1 R_2^2}{R_2^2 + \omega^2 L_1^2}.$$

When the currents in the two branches are equal, the maximum of the current delivered to the two branches will be

$$I = \frac{E_0}{R_2 \sqrt{2}}$$

Also $tan\ \phi = 1$

$$R = \frac{R_2}{2} \text{ and } L = \frac{L_1}{2}$$

163. *Graphical representation of a periodic current in a circuit having resistance and capacity.*

Equations (325) and (326) determine the current in a circuit having resistance and capacity where the *E. M. F.* is represented by Eq. (298). These three equations may be written

$$E = E_0 \sin \omega t \text{......................} (298)$$

$$i = I \sin (\omega t + \phi) \text{..................} (325)$$

where

$$I = \frac{E_0}{\sqrt{R^2 + \dfrac{1}{\omega^2 C^2}}}$$

$$tan\ \phi = \frac{1}{\omega R C} \text{......................} (326)$$

The equation for I may also be written

$$E_0^2 = R^2 I^2 + \frac{I^2}{\omega^2 C^2}$$

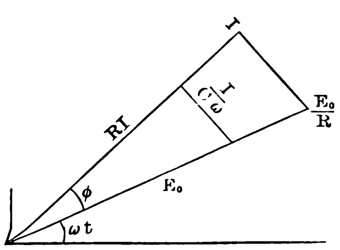


Fig. 117.

This equation like the corresponding one for a circuit having resistance and capacity, may be represented by the sides of a right-angled triangle, of which E is the hypotenuse, and $R\ I$ and $\frac{I}{C\omega}$ are the sides. The term R I represents the part of

the impressed $E.\ M.\ F.$ which maintains the current in the resistance R and $\frac{I}{\omega C}$ represents the part which is required to balance the back $E.\ M.\ F.$ of the condenser. If ϕ represent the angle of advance of I on E_0 then in Fig. 117 we have

$$tan\ \phi = \frac{1}{\omega R C}$$

which satisfies (326).

Dividing the side $R\ I$ by R we obtain a vector I which will be greater or less than $R\ I$ as R is less or greater than unity. As the triangle revolves about the point of the angle ϕ with an angular velocity ω, the projections of E_0 and I on the vertical axis are continually represented by equations (298) and (325). The difference between this case and the one representing a circuit having resistance and self-induction as shown in Figs. (105) and (106), is that the vector I is in advance of E_0 in this case, while there it was behind E_0.

The general equation for $E.\ M.\ F.$ in a circuit having resistance and capacity is given in (282) which is

$$E = Ri + \frac{Q}{C}.$$

Substituting for E, i and Q the values which characterize a periodic current, we have from (298), (325) and (338),

$$E = E_0\,sin\ \omega t = R\ I\,sin\,(\omega t + \phi) - \frac{I}{\omega C}\,cos\ (\omega t + \phi)\ (355).$$

This equation shows that the two terms of the final member differ in phase by 90°, since they involve *sin* and *cos* of the same angle. $E_0\ sin\ \omega t$ is the projection of E_0 on the vertical axis of Fig. 117. This projection is evidently equal to the difference between the projections of the other two sides, which is what the last equation asserts when ωt is less than 90°.

If ωt be made 90° the last equation will become,

$$E_0 = R\ I\ cos\ \phi + \frac{I}{\omega C}\,sin\ \phi$$

element at right angles to the lines of force is the same as that urging dm, for it is cut by the same number of lines during a small motion in arc. The force thus urging this projection carrying the current i is by Eq. (221) $H\ i\ d\ m\ sin\ \theta$.

The tangential component of this force is $H\ i\ d\ m \times sin\ \theta\ cos\ (\omega t - 90°) = H\ i\ dm\ sin\ \theta\ sin\ \omega t$. Since $dm = R'd\ \theta$ this may be written $H\ i\ R'\ sin\ \omega t\ sin\ \theta\ d\ \theta$. Substituting the value of i from (378) we have for this tangential force

$$dt = H\ R'\ I\ sin\ (\omega t - \phi)\ sin\ \omega t\ sin\ \theta\ d\ \theta.$$

It is evident that this force will be zero four times during a revolution, viz.: when $\omega t = 0$ or 180 in which cases $E = 0$, and when $wt = \phi$, or $\phi + 180$ in which cases i will be zero. This is due to the fact that we have assumed that the current lags on the $E.\ M.\ F.$ During the interval when the signs of E and i are unlike this force will be negative. The wire will be urged as a motor in the direction it is going.

The work done in moving the wire through an angle ωdt, in which case the force dT is exerted over an arc $r\ \omega\ d\ t = R'\omega\ sin\ \theta\ d\ t$ is

$$d^2w = H\ R'^2 I\ sin\ (\omega t - \phi)\ sin\ \omega t\ \omega dt\ sin^2\ \theta\ d\theta.$$

If we integrate this in θ between 0 and $2\ \pi$ we shall have the work done in turning the wire through the angle ωdt. We shall then integrate in ωt between 0 and π. This work of turning the wire through 180° is, therefore,

$$w = H\ R'^2\ I \int_0^{\pi} sin\ (\omega t - \phi)\ sin\ \omega t\ \omega dt \int_0^{2\pi} sin^2\ \theta\ d\ \theta.$$

The integral in θ is equal to π (see section 130).

Expanding the other function it becomes

$$cos\ \phi \int_0^{\pi} sin^2\ \omega t\ \omega dt - sin\ \phi \int_0^{\pi} sin\ \omega t\ cos\ \omega t\ \omega dt.$$

Of these integrals the first is $\frac{\pi}{2}$ and the second is zero.

In Fig. 107 the angle α represents $\omega\, t$, and it is evident that the two terms of the last equation are the projections of the sides $R\,I$ and $L\,\omega\,I$ on the axis y. The component $\omega\,L\,I$ is shown by the last equation to be advanced in phase 90° ahead of the component $R\,I$ and this is also shown in the figure. The effective *E. M. F.*

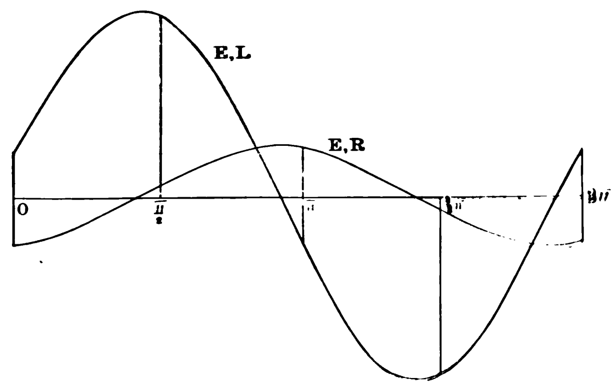


Fig. 108.

Components of *E. M. F.* which balance self-induction and maintain the current through the resistance.

due to one component will be zero when that due to the other is at its maximum.

The component $\omega\,L\,I$ is that part of the impressed *E. M. F.* which balances the back *E. M. F.* due to the self-induction of the coil, and the component $R\,I$ is the part which maintains the current through the ohmic resistance. The *E. M. F.* which maintains the current is represented by a vector whose direction is the same as that of the vector representing I. The *E. M. F.* which balances the self-induction of the coil is represented by a vector at right angles to the former and is equal and opposite to the vector which represents the back *E. M. F.* due to the self-induction of the coil.

We shall apply Eq. (245) to the conditions in *Figures 100 and 101, in which* $R =$

the horizontal diameter $E_0\,\omega\,C$ remains constant. The current I will increase to $E_0\,\omega\,C$ when $R = 0$ and the angle of advance of I will increase to 90°. In the corresponding case in self-induction the current and angle of lag were affected in this way by a decrease of R. (See Figs. 109 and 110.)

On the other hand if ω or C be increased in the case represented by Fig. 118, the effect on I and on the angle of advance is the same as would result by decreasing ω or L in Figs. 109 or 111. In the former case increasing ω or C decreases the angle of advance and increases the current. In the latter case increasing R or L increases the angle of lag and decreases the current. In both cases decreasing R increases the current, but in Fig. 118 this will increase the angle of advance while in Fig. 109 it will increase the angle of lag.

By (326) it is evident that if the angle of advance is to remain constant while R and C vary, the product $R\,C$ must remain constant. By equation (302) if the angle of lag is to remain constant while R and L vary, the ratio $\frac{L}{R}$ must be kept constant. Similar relations involving ω and any of the other quantities will be apparent on inspection of these two equations.

165. *Conductors having resistances and capacity in series.*

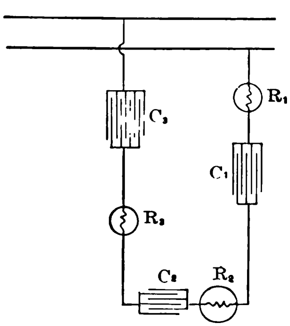


Fig. 119.
***Resistances* and Capacities in Series.**

Suppose the maximum current be given, and we are required to find what must be the maximum *E. M. F.*, E_0, which will maintain the current through resistances and capacities $R_1 C_1$; $R_2 C_2$, etc., in series with each other. In Fig. 120 draw the line $\overline{O\,I}$ to represent the maximum current flowing. From O lay off $R_1 I$ and at its end erect a perpendicular $\frac{I}{C_1\omega}$. Close this triangle by a diagonal. It will represent

the value E'_0 required to maintain the current through the first resistance R_1 with a capacity C_1. Lay off on $\overline{OI}$ an aditional length representing R_2I and at its end erect a perpendicular $\frac{I}{C_1\omega} + \frac{I}{C_2\omega}$. Then R_2I is the $E.\ M.\ F.$ required to maintain the current in the wire of resistance R_2 and $\frac{I}{C_2\omega}$ is the part of the impressed $E.\ M.\ F.\ E'_0$, required to balance the condenser potential. The

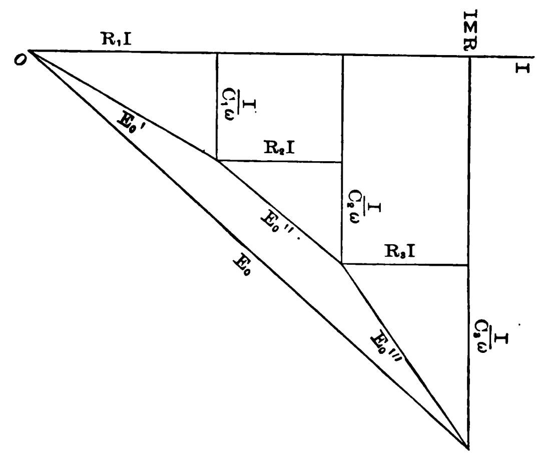


Fig. 120. Resistances and capacitties in series.

$E.\ M.\ F.\ E''_0$, is represented by the diagonal of the triangle of R_2I, and $\frac{I}{C_2\omega}$ is the impressed $E.\ M.\ F.$ required for the second resistance and capacity. Proceed in this manner until all the capacities and resistances have been included. The diagonal E_0 which closes O with the remote end of the last diagonal, will, be the value of E_0 required.

The circuit having one resistance and a single condenser which will be equivalent must have an $E.\ M.\ F.$ triangle whose sides are $R\,I = I\,\Sigma\,R$ and $\frac{I}{\omega C} = \frac{I}{\omega}\,\Sigma\,\frac{1}{C}$.

The resistance must be the sum of the separate resistances, and the reciprocal of the equivalent capacity must be the sum of the reciprocals of the separate capacities. The angle of advance would be determined by the equation

$$tan\ \phi = \frac{1}{\omega \Sigma R \Sigma C}$$

If the triangle Fig. 120 be revolved about O as has been explained, the simultaneous projections of I and E_0 will give the values of i and E at any instant, in accordance with equations (325) and (298).

If the impressed $E.\ M.\ F.$ be given, and we wish to find the current through a series of condensers and resistances, we may proceed as follows. Assume any current I, and find as has been just explained, the impressed E_0 which would produce this current in the given system. Then will

$$\frac{\text{Assumed } I}{\text{Computed } E_0} = \frac{\text{required } I}{\text{given } E_0}$$

Suppose we have resistances $R_1 = 0.1$, $R_2 = 0.2$ and $R_3 = 0.4$ and the current maximum is $I = 20$ amperes. The amperemeter reading would be $\frac{20}{\sqrt{2}}$ amperes, this being the virtual current. The resistance of the equivalent coil would be 0.7 ohm, and the $E.\ M.\ F.\ E'_0$ required to maintain this current through the resistance of the circuit is $E'_0 = I\,\Sigma\,R = 14$ volts. The power delivered to the system and used in heating the resistances is $\frac{1}{2}\,I^2\,\Sigma\,R = 140$ watts.

The $E.\ M.\ F.$ required to balance the potential of the equivalent condenser would be $E_0'' = \frac{I}{\omega}\,\Sigma\,\frac{1}{C}$. Suppose $\omega = \frac{2\,\pi}{T} = 800$ radians per second, $C_1 = 0.00005$, $C_2 = 0.00007$, $C_3 = 0.0001$ farad. Then the tangent of the angle of advance of I on E_0 will be

$$\frac{1}{\times 0.7 \times 0.00022} = 8.11$$

$$= 82°58'$$

The E. [illegible]
would [illegible]

The [illegible]
[illegible]

The vectors I_1 and I_2 can be then drawn, since their direction is known. The chords $R_1 I_1$, $R_2 I_2$ of the semicircle are determined by the equations,

$$E'_0 = R_1 I_1 = E_0 \cos \phi_1$$
$$E''_0 = R_2 I_2 = E_0 \cos \phi_2 \text{.............}(357).$$

These values E'_0 and E''_0 represent the part of the E.

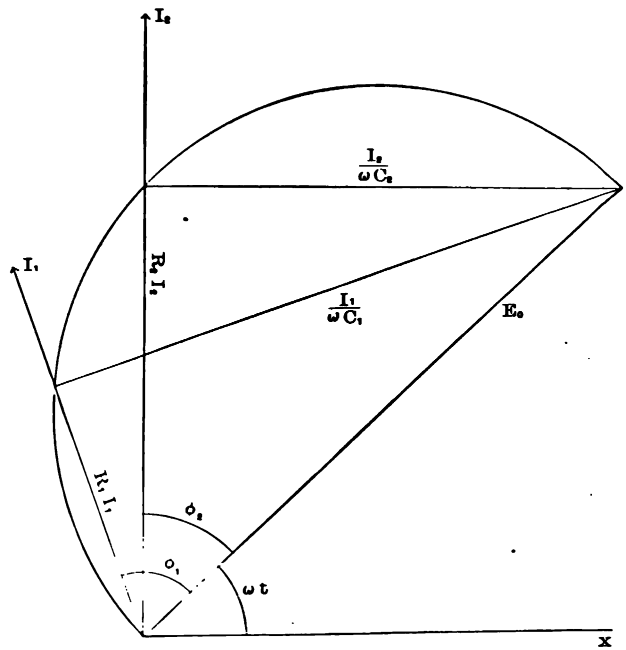


Fig. 122.
Resistances with capacities in multiple.

M. F. required to balance the condensers. We have, therefore, in known terms

$$I_1 = \frac{E_0}{R_1} \cos \phi_1$$
$$I_2 = \frac{E_0}{R_2} \cos \phi_2 \text{..............................}(358).$$

The projections of I_1 I_2 and E_0 on the vertical axis Fig.

123 represent at any instant the currents in the branches, and the periodic *E. M. F.* We have for these currents

$$i_1 = I_1 \sin(\omega t + \phi_1)$$
$$i_2 = I_2 \sin(\omega t + \phi_2)$$

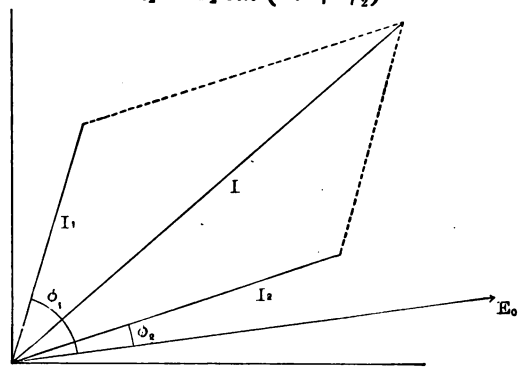


Fig. 123.
Resultant of component currents.

The sum of these currents will represent the current in the undivided part of the circuit, or

$$i = i_1 + i_2 = I_1 \sin(\omega t + \phi_1) + I_2 \sin(\omega t + \phi_2).$$

Since the sum of these projections, is equal to the projection of I, which is the trigonometric sum of I_1 and I_2 we have

$$i = I \sin(\omega t + \phi) \dots\dots\dots\dots (359)$$

where ϕ is the angle of advance of the vector I. This equation has the same form as (349). In order to determine I and ϕ from Fig. 123,

$$I = \sqrt{I_1^2 + I_2^2 + 2\, I_1\, I_2 \cos(\phi_1 - \phi_2)} \dots\dots\dots (360).$$

which gives I in known terms.

Also

$$\cos(\phi - \phi_2) = \frac{I_2^2 + I^2 - I_1^2}{2 I_2 I}$$

which by (360) becomes

$$\cos(\phi - \phi_2) = \frac{I_2 + I_1 \cos(\phi_1 - \phi_2)}{\sqrt{I_1^2 + I_2^2 + 2 I_1\, I_2 \cos(\phi_1 - \phi_2)}} \dots (361)$$

which determines ϕ.

Knowing ϕ we may determine the resistance of the

equivalent circuit. From the *E. M. F.* triangle (Fig. 122) we have

$$R\,I = E_0 \cos\phi$$

$$\therefore R = \frac{E_0}{I}\cos\phi \quad \text{.......................(362)}$$

and for the capacity of the same circuit we have

$$\frac{I}{\omega C} = E_0 \sin\phi$$

$$\therefore C = \frac{I}{\omega\, E_0 \sin\phi} \quad \text{.......................(363).}$$

Example. Suppose the maximum potential on the ends of the divided circuits to be $E_0 = 120$ volts. Let $\omega = 800$ radians per second, and for two branches let $R_1 = 0.1$ ohm, $R_2 = 0.5$ ohm, $C_1 = 0.00078$ farad, $C_2 = 0.00052$ farad. Then by (356)

$$\tan\phi_1 = \frac{1}{800 \times 0.1 \times 7.8 \times 10^{-4}} = 16.02$$

$$\therefore \phi_1 = 36°26'$$

$$\tan\phi_2 = \frac{1}{800 \times 0.5 \times 5.2 \times 10^{-4}} = 4.80$$

$$\phi_2 = 78°14'$$

This determines the direction of vectors I_1 and I_2. The chords R_1I_1 and R_2I_2 are then by Eq. (357)

$$E'_0 = R_1I_1 = 120 \cos 86°26'$$

$$E''_0 = R_2I_2 = 120 \cos 78°14'$$

$$\text{or } E'_0 = 7.465 \text{ volts.}$$

$$E_0'' = 24.471 \text{ volts.}$$

By (358)

$$I_1 = \frac{7.465}{0.1} = 74.65 \text{ amperes.}$$

$$I_2 = \frac{24.471}{0.5} = 48.94 \text{ amperes.}$$

By (361) we have as in the similar case in section 162

$$I = 123.29 \text{ amperes.}$$

Similarly by (360) and (362) we have as in section 162

$$\phi = 83°\,14' \qquad R = 0.1147 \text{ ohm.}$$

By Eq. (363)

$$C = \frac{123.29}{800 \times 120 \sin 83°14'}$$

$$= 0.00129 \text{ farads.}$$

The current in the undivided branch is then at any instant

$$i = 123.29 \sin (\omega t + 83°14')$$

and the currents in the two branches will be

$$i_1 = 74.65 \sin (\omega t + 86°26')$$
$$i_2 = 48.94 \sin (\omega t + 78°14').$$

The equivalent circuit being thus determined, it may be combined with a third, and this resultant may be combined with a fourth, until all the multiple circuits are involved. The final equivalent may then be combined with any number of sections or parts of a circuit in series with the divided circuits any of which may be equivalents of more complex systems. The method of doing this has been fully explained.

167. *Special Cases.*

If as in section 162 we have a divided circuit, the currents in the two branches will be

$$i_1 = \frac{E_0}{\sqrt{R_1^2 + \omega^2 L_1^2}} \sin (\omega t - \phi_1)$$

where $\tan \phi_1 = \frac{\omega L_1}{R_1}$

$$i_2 = \frac{E_0}{\sqrt{R_2^2 + \omega^2 L_2^2}} \sin (\omega t - \phi_2)$$

where $\tan \phi_2 = \frac{\omega L_2}{R_2}$.

If the resistance in the first circuit be $R_1 = 0$ and the self-induction in the second circuit be $L_2 = 0$, these equations become

$$i_1 = \frac{E_0}{w L_1} \sin (\omega t - 90°)$$
$$\tan \phi_1 = \infty$$
$$i_2 = \frac{E_0}{R_2} \sin \omega t$$
$$\tan \phi_2 = 0.$$

The second circuit will then be in unison with the impressed *E. M. F.* and the first circuit would lag 90°.

The average and likewise the virtual values of these

either circuit, since the potentials at these points are equal. The equation for the four currents will be

$$i_1 = I \sin (a - \phi)$$
$$i_2 = - I \cos (a - \phi)$$
$$i_3 = - I \sin (a - \phi)$$
$$i_4 = I \cos (a - \phi)$$

The *E. M. F.* in each circuit will be

$$E_1 = E_0 \sin a$$
$$E_2 = - E_0 \cos a$$
$$E_3 = - E_0 \sin a$$
$$E_4 = E_0 \cos a$$

The power required to drive the armature against the electro-magnetic forces will be as in section 177

$$\frac{dw_1}{dt} = E_0 I \sin (a - \phi) \sin a$$
$$\frac{dw_2}{dt} = E_0 I \cos (a - \phi) \cos a$$
$$\frac{dw_3}{dt} = E_0 I \sin (a - \phi) \sin a$$
$$\frac{dw_4}{dt} = E_0 I \cos (a - \phi) \cos a.$$

The total power at any instant is then

$$\frac{dw_1}{dt} + \frac{dw_2}{dt} + \frac{dw_3}{dt} + \frac{dw_4}{dt} = 2 E_0 I \cos \phi.$$

This may be written $\frac{4}{2} E_0 \ I \cos \phi$. The power is the same for each current as in the case of triphased currents, as was to have been expected, but of course the values of E_0 and I will have different numerical values from those of the same armatures used in triphased service.

It may be observed that if two of the coils which are opposite each other be suppressed, as for instance coils 2 and 4 of the last equations, we shall have for the power at any instant

$$\frac{dw_1}{dt} + \frac{dw_3}{dt} = 2 E_0 I \sin (a - \phi) \sin a.$$

This equation is identical with Ei in (310) since R in the circuit which that equation represents, is identical with $2R$ in the present case.

CHAPTER X.

IRON CORES IN ELECTRO-MAGNETIC SYSTEMS.

182. *Ballistic Galvanometer.*

As preliminary to the study of the modifying effects of iron and other magnetic cores we may examine the instruments used in the study of these effects.

If an instantaneous discharge of electricity flows through the coil of a galvanometer having a heavy needle, the first throw of the needle may be used to meas-use the quantity of the discharge.

The kinetic energy of a rotating body having a moment of inertia I and an angular velocity a is $w = \frac{1}{2} a^2 I$. Then $dw = I\, a\, d\, a$, at any instant during discharge. Let a' represent the mean radius of the coil windings, and n the number of windings, the current at any instant being i. Then the strength of the field in which the needle is placed may be taken as $\frac{2\,\pi\, n\, i}{a'}$.

The moment of the magnetic needle being $l\, m = M$ the moment of the force acting on the undeflected needle is $\frac{2\,\pi\, n\, i}{a'}$ M. Call this quantity $B\, i$. Then

$$I\, a\, d\, a = B\, i\, a\, dt$$

or

$$I\, d\, a = B\, i\, dt.$$

If we suppose the discharge to take place before the needle has moved, then we may integrate this equation. Calling $\int i\, dt = q$, the total quantity passing during the momentary discharge we have

$$I\, a = B\, q$$

$$\text{or } a = \frac{B}{I}\, q \quad \text{......................(384).}$$

Since B and I are constants depending upon the galvanometer, we find that such a momentary discharge, which is finished before the needle of the galvanometer can appreciably move, will produce an angular velocity directly proportional to the quantity of electricity discharged. This result is evidently independent of the law of discharge. The resulting motion of the needle is determined by the integrated effects of the entire operation.

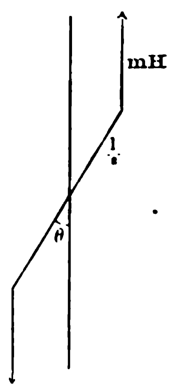


Fig. 131.

Let θ = the angle of final deflection of the needle, and l the length of the needle. The work done against the earth's pull of $H\,m$ on each of the magnet poles is

$$m\,H(1-\cos\theta)\,l$$
$$=MH(1-\cos\theta)=2\,M\,H\sin^2\frac{\theta}{2}.$$

Then the total energy imparted to the body or $\frac{1}{2}I\,a^2$ is consumed in producing this deflection against the earth's resisting moment, or

$$\frac{1}{2}I\,a^2=2\,M\,H\sin^2\frac{\theta}{2}$$

$$\therefore a=2\sin\frac{\theta}{2}\sqrt{\frac{M\,H}{I}}=\frac{B}{I}q.$$

We have then for the quantity of discharge

$$q=\frac{2\sin\frac{\theta}{2}}{B}\sqrt{M\,H\,I}$$

Restoring the value of B

$$q=\frac{a'H}{\pi n}\sin\frac{\theta}{2}\sqrt{\frac{I}{M\,H}}.$$

If t = the time of semi-vibration of the needle under the influence of the earth we have by (161)

$$t=\pi\sqrt{\frac{I}{HM}}.$$

Putting the value of this radical in the last equation we have

$$q = \frac{a' H t}{\pi^2 n} \sin \frac{\theta}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (385).$$

Example. Suppose the average radius of the windings to be $a' = 4$ centimetres, the strength of the earth's horizontal component to be $H = 0.2$, the time of a semi-vibration of the needle under the action of H to be 5 seconds, the number of windings of the coil to be 100. A beam of light reflected from a mirror on the needle is thrown upon the wall 600 centimetres from the centre of the needle. When the discharge takes place, the spot of light moves 15 centimetres. The circumference having a radius 600 is 1200 π. The angle θ is, therefore, $\frac{15}{1200\,\pi}\, 360° = 1°26.'$ Hence $\frac{\theta}{2} = 0°43'$ and $\sin \frac{\theta}{2} = 0.0123$. Therefore

$$q = \frac{4 \times 0.2 \times 5}{\pi^2\, 100}\, 0.0123$$
$$= 0.0000497\ C.\ G.\ S.\ \text{units}$$
$$= 0.000497\ \text{culombs}.$$

A glow lamp requiring a current of an ampere, requires one culomb of electricity to be forced through it each second.

Equation (385) from which this computation has been made should not be relied upon for precise results, since the average strength of the field due to the coil which is occupied by the needle, is considerably greater than $\frac{4\,\pi\, n\, i}{a}$ (see section 115). The equation serves only as a useful guide in designing a ballistic galvanometer, and in enabling us to understand the general principle of its operation.

183. *Calibration of the scale of the ballistic galvanometer.*

The calibration of the scale of the ballistic galvanometer may be effected by means of a flat open coil of rather *large diameter* which admits of sudden rotation *through 90° or 180°* around a diameter. Such a coil *called an earth inductor*.

$\frac{4\pi S i}{l}$, where S is the number of windings on the length l. If we multiply this by the cross section A of the cylinder we have

$$N = \frac{4\pi s i}{\frac{l}{A}} \quad \text{.......................} \quad (387).$$

The numerator of this expression is the work required to carry unit pole along any closed path which links with the S windings back to the starting point (section 113). It is the difference of magnetic potential due to the helix. It is called the magneto-motive force of the magnetic circuit. The denominator $\frac{l}{A}$ represents the magnetic resistance, or reluctance of the air column whose length is l and whose section is A. This equation is the same as Ohm's law. This equation represents that the number of lines passing through the central plane is the same as would be produced by a magneto-motive force $4\pi S i$ on the terminals of an air column whose length is l and whose section is A. We are justified in using this equation for the reason that the earth inductor gives the same value of N that we get by this means.

The wire of the helix is slightly parted at the middle to allow the winding of a narrow coil of fine wire upon the cylinder. This coil is to take the place of the earth inductor of the previous section.

We will now determine the windings of a coil and ballistic galvanometer that will give an angle of throw of 45° when the field of the helix is reversed. This field has a number of lines $N = 43.43 \times \frac{1}{4}\pi \times 100$ or 3411 lines in all. When the current is reversed these lines will all disappear and will be threaded through in the opposite direction. We shall assume that the field occupied by the needle is that at the centre of the galvanometer coil, so that our galvanometer will be somewhat more sensitive then we have assumed. Then from equations (385) and (386) we have

$$\frac{N n'}{R \sin\frac{\theta}{2}}.$$

We have assumed $\theta = 45°$ and we may assume $t = 5$ seconds. $H = 0.2$. We may also assume $a' = 3$ centimetres as being a desirable dimension for the average radius of the windings of the galvanometer. In accordance with what precedes we take $N = 2 \times 3411 = 6822$.

$$\therefore \frac{R}{n\,n'} = \frac{\pi^2\, 6822}{0.2 \times 5 \times 3 \times \sin 22°30'}$$
$$= 58650.$$

We shall make the resistances of the test coil and that of the galvanometer coil equal, the sum of the two being R ohms. We shall use for both coils, No. 30 *B. W. G.* the diameter of which is 0.30 millimetre. The resistance of this wire per running centimetre is 0.00254 ohm $= 2.54 \times 10^6$ *C. G. S.* units. The diameter of the average winding in the test coil may be about 10.5 which we will assume as a first approximation. Then

$$(10.5\,\pi\, n' + 6\,\pi\, n)\ 2.54 \times 10^6 = R$$
$$10.5\ n' = 6n$$
$$\therefore \frac{12\,\pi\, n \times 2.54 \times 10^6 \times 10.5}{6n^2} = 58650$$
$$n = 286 \quad n' = 163.$$

Having now learned how many windings must be disposed of in the space which should be used, we may if necessary assume the average radius of the windings to be such as will more suitably accommodate the wires, and recompute n' and n as before.

After the apparatus has been made, the constant $c = \dfrac{N\, n'}{R \sin \frac{\theta}{2}}$ must be determined by actual experiment, in which N may be varied and the corresponding value of θ is to be measured. We can then use the apparatus for determining N in other cases yet to be discussed. We shall thus have

$$N = \frac{c\, R}{M} \sin \frac{\theta}{2} \text{.................. (388).}$$

or

$$N n' = c\, R \sin \frac{\theta}{2} \text{..................... (389).}$$

Knowing $c\,R$ we may use coils C C' which are similarly excited by a known current which can be reversed. If N represent the lines carried by the iron core and n' the windings in coil C' we have by (388)

$$N' = \frac{c\,R}{n'}\, sin\, \frac{\theta}{2}.$$

It is found that N is the same whether coils C consist of a few turns of wire carrying a large current, or many turns of wire carrying a small current, so long as the number of ampere-turns in C remains unchanged. But if the coils C occupied the entire length of the test specimen the number of lines threading through C' would be by (387)

$$N' = \mu\, \frac{4\,\pi\, S\, i}{\frac{l}{A}} = \frac{4\,\pi\, S\, i}{\frac{l}{\mu\, A}} \ldots\ldots\ldots\ldots\ldots (391)$$

where S is the number of windings in coils C, and l is the length of the core. Experiment with an exploring needle shows that there is no appreciable leakage of magnetic lines from any part of the iron, so that the lines threading through C' find their way through the iron circuit because of its great permeability compared with air. This explains why coils C need not extend along the entire length of the specimen, and why (391) and (387) hold true practically even when there are only a few turns of wire. In such cases l of (391) becomes the length of the iron core instead of the length of the coil. Since N', S, i, l and A of (391) can all be measured, by means of the apparatus of Fig. 132, it is evident that μ can be determined.

It appears that μ in Eq. (391) corresponds to specific magnetic conductance and that $\frac{l}{\mu\, A}$ represents magnetic resistance or reluctance. If we divide (391) by A, we have for the number of magnetic lines per square centimetre of cross-section of the iron core

$$B = \frac{N'}{A} = \frac{4\,\pi\, S\, i}{l}\, \mu \ldots\ldots\ldots\ldots\ldots (392).$$

For t' of field when no iron core is preser

in coils C, we have from (387) if the coils have a length l, equal to that of the core,

$$H = \frac{N}{A} = \frac{4\pi S i}{l} \text{.....................} (393).$$

From (392) and (393)

$$B = \mu H.$$

This equation is identical with (156). The magnetic lines are practically all confined to the iron circuit in the apparatus shown in Fig. 132. The number of lines carried by the iron and due to the magnetizing coil depends on the number of ampere turns without reference to the length of the coil. The l of equations like (392) is the length of that part of the magnetic circuit which offers reluctance to the passage of magnetic lines. It is evident that l may have this meaning also in such equations as (188), or (393) since the reluctance in a long helix is practically that of the air column whose length is l. These considerations also show that the iron core to be tested may be formed into a closed ring, upon which the magnetizing coils C and the test coil c' may be wound. The length l then becomes the average circumference of the iron, or the length of its middle fibre.

Eq. 392 is continually used in the designing of dynamos and motors. It appplies to any part of a magnetic circuit, in the same way that Ohm's law applies to any part of an electric circuit. For example the number of ampere turns required to force B lines per square centimetre through the two air gaps of a dynamo may be determined as follows:—

Let the area of one pole-face be 1600 square centimetres. Let the distance from pole-face to the iron core be 1.5 centimetre on each side. The total magnetic lines required by the armature in order that it may be able to develop the required $E. M. F.$ at the desired speed may be $N = 9{,}600{,}000$. This would require 6,000 lines per square centimetre of cross-section of the air-gap. The value of μ being unity for air we have by (392)

$$Si = \frac{2 l B}{4\pi} = \frac{2 \times 1.5 \times 6000}{4\pi} = 14\text{..}$$

Since the measurements have all been

the number of ampere turns required on the field in order to maintain 6,000 lines per square centimetre through the air gaps is to be found by converting the *C. G. S.* units of current into amperes. We thus have 1430×10 ampere turns. Let the average length of path of the lines through the iron drum of an armature be 13 centimetres, and the average section through which they flow be taken as 800 square centimetres. Assume $B = 12000$ lines per square centimetre of iron. For this value of B, μ is about 2500 for soft iron. Hence the number of ampere turns required for this part of the circuit becomes

$$Si = \frac{10 \times 12{,}000 \times 13}{4 \times 2500 \times \pi} = 49.6.$$

In this way each part of the magnetic circuit may be treated, and the sum of the number of ampere turns for the separate parts determined*. This will determine the exciting power in ampere turns which must be put on the field coils. For high degrees of magnetization leakage from the sides of the iron cores must, however, be dealt with. Practical methods of doing this may be found in works on the designing of electrical machinery.

185. *Permeability, Magnetometer method.*

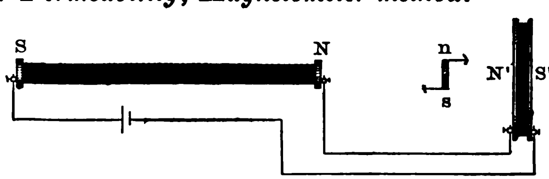


Fig. 138.

Long iron specimen in *N. S.* deflecting a needle n. s. Coils balanced against each other.

A long helix *N. S.* having a movable iron core may be used as the deflecting magnet of a magnetometer. The effect of the helix should be compensated by an opposing helix *N′S′* which is preferably energized by the same current.

* See Thompson's Dynamo Electric Machinery. Chapters VII and XVI, 4th ed.

We may then determine the magnetic moment of the iron core.

If the magnetometer needle be deflected over an angle a, the earth's field being $H_e^{\prime}$, then the moment of the iron core is by (167), neglecting the small term,

$$M = \frac{1}{2} H_e r^3 \tan a.$$

For the moment per unit volume of the core, calling v the volume

$$I = \frac{M}{v}.$$

This value of I is identical with that of Eq. (155).

If the current in the helix be gradually increased, the field F (or H) occupied by the iron core will increase in strength, and M and I will also increase. F (or H) is computed from (188) and M and I from the above equations. In this apparatus the length of the iron core should be at least four hundred diameters. With iron a quarter of an inch in diameter, this requires a rod one hundred inches long. The helix should cover the entire length.

An helix having a length of 254 centimetres was wound on a brass tube having an external diameter of 0.963 centimetre. This helix was made up from wire whose diameter was 0.217 centimetre. The diameter of the covered wire was 0.2286 centimetre. The distance from centre to centre of the wire measured across a diameter was, therefore, 1.192 cm. The value R^2 of Eq. (184) was, therefore, 0.36 and $\frac{l}{2} = 127$. The wire was wound in a single layer on the tube, 1095 windings covering 254 cm. When the current is 10 amperes or 1 $C. G. S.$ unit, the value of H at various distances D from the centre of the coil may be computed from (184) which becomes

$$H = 2\pi \frac{1095}{254}\left[\frac{D+127}{\sqrt{(D+127)^2+0.36}} - \frac{D-127}{\sqrt{(D-127)^2+0.36}}\right]$$

This force does not appreciably vary until $D = 120$ cm., when it begins to fall off. When $D = 126$ it is, however,

50.4, and drops to 27.07 when $D = 127$. This is exactly half the value at the centre where $D=0$. An iron core placed in this helix is, therefore, all practically in a field whose strength is 54.14, when a current of ten amperes is passing through the wire. The value of H, or F, in terms of D is shown in Fig. 134. Starting with an iron core which has never been magnetized, a small current i, is passed through the wire coil. The value of H is computed for this current. The angle of deflection of the magnetometer needle is read and M is then computed. The dimensions of the core being known, its volume in cubic centimetres is known and $I=\frac{M}{V}$ is determined. The magnetic susceptibility $K=\frac{I}{H}$ is then determined. (See Eq. 155.)

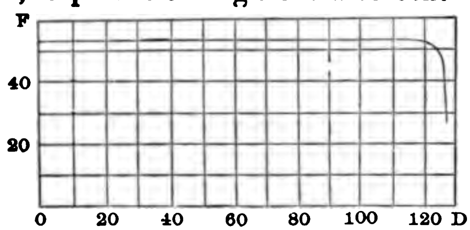


Fig. 134.
Field along the axis of a long helix. $D =$ distance from centre.

The value of B is then determined from (156) as likewise the value of μ or the magnetic permeability.

Example. The long helix just described was provided with a core of soft iron whose length was 254 cm. and cross-section was 1.267 sq. cm. The volume of the core was 322 c.c. A current of 2.50 amperes was passed through the coil. The deflection of the magnetometer needle was 2°21′. The strength of the earth's field was 0.20. The distance r from centre to centre of the two magnets was 15 feet or $15 \times 12 \times 2.54$ centimetres. Hence the moment of the core was

$$M = \frac{1}{2} 0.20\,(15 \times 12 \times 2.54)^3 \tan 2°21' = 392{,}200.$$

The intensity of magnetization was

$$I = \frac{392200}{322} = 1218.$$

The value of H was one-fourth of that represented in

Fig. 134, or $H = \dfrac{54.14}{4} = 13.54.$

The magnetic susceptibility was, therefore,

$$K = \frac{1218}{13.54} = 89.$$

The magnetic induction within the iron was

$$B = 13.5 + 4\,\pi \times 1218 = 15300.$$

The magnetic permeability was

$$\mu = \frac{15300}{13.5} = 1130.$$

The following table gives the results of determinations by Ewing upon soft iron wire. The determinations were made with a rising current:

H	I	K	B	μ
0.0	0	——	0	——
0.32	3	9	40	120
0.84	13	15	170	200
1.37	33	24	420	310
2.14	93	43	1,170	550
2.67	295	110	3,710	1,390
3.24	581	179	7,300	2,250
3.89	793	204	9,970	2,560
4.50	926	*206*	11,640	*2,590*
5.17	1,009	195	12,680	2,450
6.20	1,086	175	13,640	2,200
7.94	1,155	145	14,510	1,830
9.79	1,192	122	14.980	1,530
11.57	1,212	105	15,230	1,320
15.06	1,238	82	15,570	1,030
19.76	1,255	64	17,780	800
21.70	1,262	58	15.870	730

By placing a small specimen of annealed wrought iron between the coned poles of an electro-magnet, much more powerful fields can be used. The specimen is surrounded by a bobbin in circuit with a ballistic galvanometer, by means of which B may be determined. H was determined by similar measurements just outside of the bobbin, the coned pole-pieces having been given an angle which produced a uniform field over a considerable region *around the* specimen.

Ewing gives the following determinations made in this way.

H	I	K	B	μ
24,000	1660	0,069	45,350	1.85

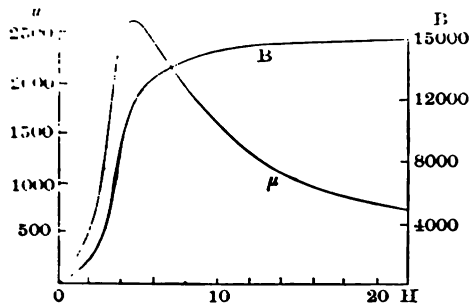


Fig. 135.

Permeability μ and induction B for fields of various strength H. (Ewing.)

While H has been made over a thousand times as strong as the greatest value in the table preceding, the magnetic moment per unit of volume has increased by only one-third of its value and is evidently approaching a limit. The magnetic susceptibility, K, has fallen from a maximum of 206 in a field 4.5 units to 0 069, and is approaching a limit zero. The permeability μ has fallen from a maximum of 2590 when the field H had a strength of 4.50 to 1.85 in a field of 24,000. The value of μ is approaching unity. Fig. 135 shows Ewing's values for μ and B in terms of H, and Fig. 136 shows the values for K and I.

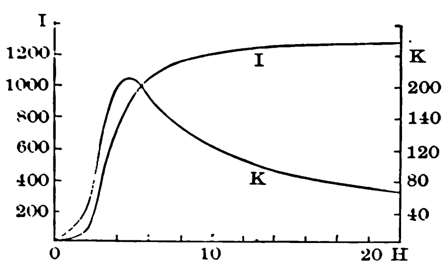


Fig. 136.

Susceptibility K and intensity of magnetization, for iron in field of strength H. (Ewing.)

186. *Magnetic hysteresis.*

If after the field has been increased in strength until a high degree of magnetization has been reached, observations like those given in the table of the previous section being made at various values of H, the operation be *reversed*, and the observations be repeated in a descend-

ing series, very different results will be obtained. After the current has been reduced to zero the value of I will still be 80 to 90 per cènt. of its value when the magnetizing field was strongest. In soft iron this magnetization is, however, very lightly held. A feeble current in the

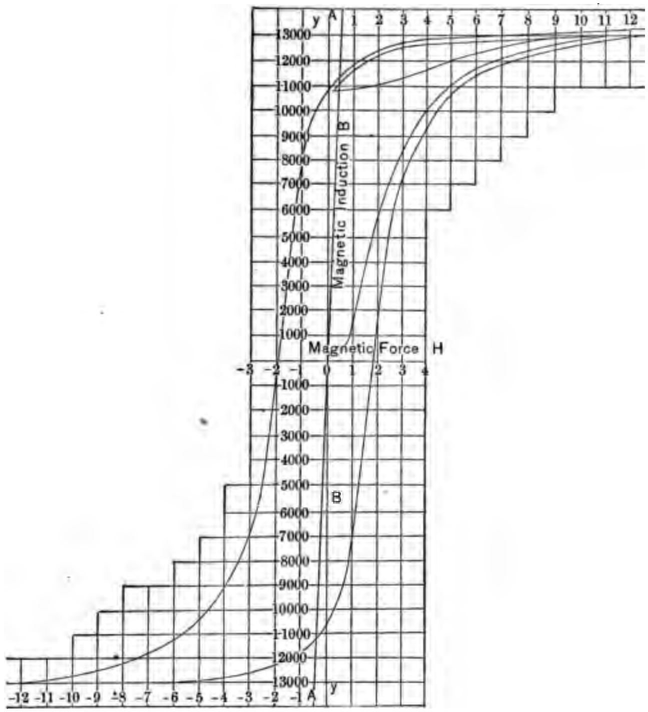


Fig. 137.
Hysteresis cycle for soft iron. (Ewing.)

reverse direction will reduce I to zero. If the reversed current be increased until it becomes as great as it was before, and then the descending series be again determined, the value I will pass through a definite cycle of change. The curve so obtained is shown in Fig. 137.* The ordinates represent the values of B and the abscissæ

* The pupil may determine values of B and H from this diagram and by equation (154) compute I. The values I plotted with the *simultaneous values of H*, will give a cycle like those shown in Fig. 138.

the value of H. The value of H varies between $+ 12$ and $- 12$. It will be seen that the value of B for any value of H, depends upon whether the magnetic strength is rising or falling. The differences may be largely obliterated by tapping or jarring the specimen during the operation. This lagging of changes in B on the change in H is called hysteresis. During the hysteresis cycle, molecular changes take place within the iron which involve a loss of energy. This energy appears as heat in the iron, and we shall proceed to show that the area of the hysteresis diagram represents this loss.

Let the iron core be in the form of a long rod as previously described, or a closed ring of length l and section A. Call n the windings per centimetre of length, so that the total windings will be ln. During any time dt, in which the current i changes by di, the increase in the number of magnetic lines per square centimetre will be $d B$. The result of this increase in B will be the starting of a counter $E. M. F.$ e in the magnetizing wire. The work of the magnetizing current in magnetizing the core in opposition to this counter $E. M. F.$ will be $\int eidt$. This integration is to be taken over a complete period, in which the current changes, say, between $+ i$ and $- i$. The total increase in the number of lines of the core in a time dt will be $A\, d\, B$ and the change per second will be $A \frac{dB}{dt}$. Since there are ln windings around these lines, the counter $E. M. F.$ induced will be

$$e = l\, n\, A \frac{dB}{dt}.$$

The power applied at any instant in magnetizing the helix is then $e\, i$, and the energy applied in a time dt is

$$\begin{aligned} eidt &= l\, n\, A\, i \frac{dB}{dt} dt \\ &= l\, n\, A\, i\, d\, B. \end{aligned}$$

The fact that this energy is independent of t, shows that the operation of magnetizing may go on as slowly as we please. We cannot diminish this loss in energy *per cycle* by diminishing the frequency of alternation. Since $l\, A$ of the last equation is equal to the volume v of

the iron core, the energy lost per cycle per unit volume is

$$\frac{dw}{dv} = n\,i\,d\,B.$$

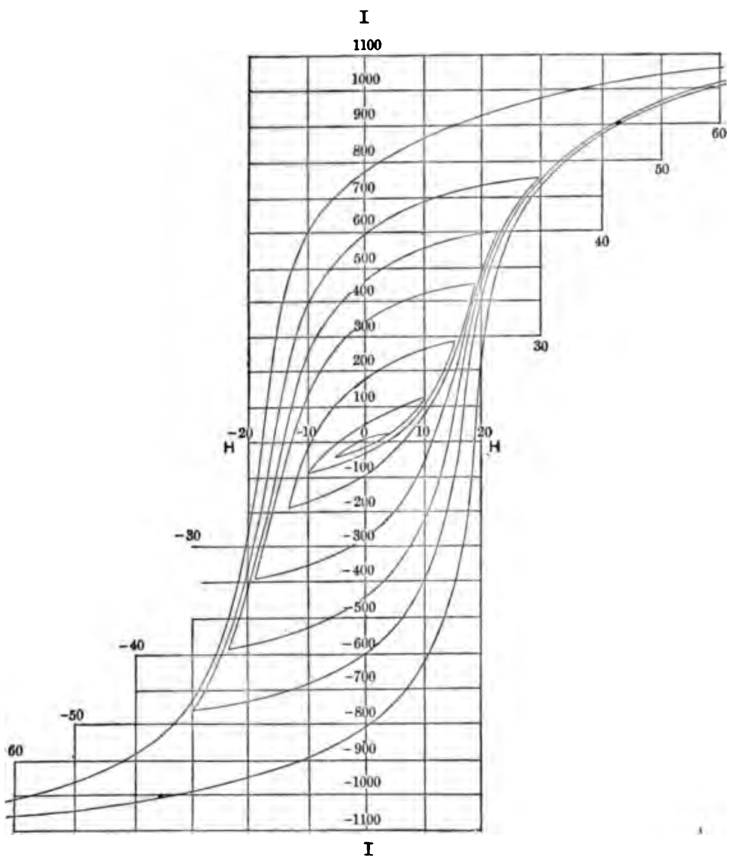


Fig. 138.
Hysteresis cycles. Piano wire.

The magnetizing force $H = 4\,\pi\,n\,i$.

$$\therefore \frac{dw}{dv} = \frac{1}{4\,\pi}\,\mathrm{H\,d\,B}.$$

Since $B = H + 4\pi I$ by (154)

$$dB = dH + 4\pi\, dI$$

and

$$\frac{w}{v} = \frac{dw}{dv} = \frac{1}{4\pi}\int H\, d\, H + \int H\, d\, I.$$

During a complete period $\int H\, d\, H$ vanishes and the resulting loss of energy in ergs per cycle is

$$\frac{w}{v} = \int H\, d\, I.$$

The value of this integral for a cycle is evidently represented by the diagrams shown in Fig. 138. It appears that the motions taking place among the particles of iron during magnetization, are not fully reversed during the descending process, when the stresses are removed. $\int F\, d\, I$ during the magnetization is greater than during the reverse operation between the same limits in I, by an amount represented by the area of the diagram between those limits. This area will depend on the extent of extreme change in H. Fig. 138 shows a series of cycles for piano wire taken from Ewing and his values for a similar series on soft iron wire are given in the following table. Columns H, B and I are the greatest values for these quantities during each cycle and correspond to the extreme ends of the pointed diagrams.* In order to draw the diagrams, each cycle must be determined by a great number of measurements similar to those previously described. Between 300 and 400 determinations were made on the ten cycles shown in the figure. The energy $\int H\, d\, I$ given in the fourth column was determined by planimeter measurements of the areas of the curves as is described in section 115.

* *Magnetic Induction in Iron and other Metals.* Ewing. Fig. 50. *p.* 106.

H	B	I	$\int H\,d\,I$ (ergs.)
1.50	1974	167	410
1.95	3830	304	1160
2.56	5950	473	2190
3.01	7180	571	2940
3.76	8790	699	3990
4.96	10590	842	5560
6.62	11480	913	6160
7.04	11960	951	6590
26.5	13720	1090	8690
75.2	15560	1230	10040

The rise of temperature of the unit volume upon which $\int H\,d\,I$ ergs of energy is applied may be determined for each of the ten cycles of the preceding table. Indicating by t the rise in temperature for the cycle and let $J =$ the number of ergs equivalent to a water-centigrade heat unit or $J = 4.18 \times 10^7$, $s =$ specific heat of iron, $= 0.11$; and $D =$ density of iron, $= 7.7$. Then

$$t = \frac{\int H\,d\,I}{J\,s\,D} \qquad \text{.........................(394).}$$

For the first cycle of the table

$$t = 2.84 \times 10^{-8} \times 410 = 0.000012.$$

It would, therefore, require a little over 80,000 cycles to cause the temperature to rise one degree. In the last cycle of the table the rise in temperature would be 0.000285, so that less than 4000 cycles would cause a rise of temperature of 1° C. from this cause.

187. *Power lost in hysteresis.*

Assuming 7.7 grammes of iron to the cubic centimetre, and 453.5 grammes to the pound, the energy dissipated in taking one pound of iron of the kind used by Ewing through a magnetic cycle is

$$w' = \frac{453.6}{7.7} \int H\,d\,I.$$

If there are n cycles per second, the power dissipated in ergs per second per pound of iron is obtained by multiplying by n.

Since 10^7 ergs per second = one watt, the power lost in watts per pound of iron becomes

$$w = \frac{453.6}{7.7 \times 10^7} n \int H \, d\, I$$

$$= 5.89 \times 10^{-6} n \int H \, d\, I.$$

Steinmetz has found a simple empirical formula, in which the energy lost in ergs per cubic centimetre per cycle may be expressed in terms of the induction density B. He finds for any cycle

$$\int H \, d\, I = y\, B^{1.6}$$

where y is a constant for a given iron. Steinmetz gives the following values for y

MATERIAL.	y	MATERIAL.	y
Very soft iron wire......	0.002	Soft annealed cast steel..	0.008
Very thin soft sheet iron.	0.0024	Soft machine steel......	0.0094
Thin good sheet iron.....	0.003	Cast steel...............	0.0120
Thick sheet iron.........	0.0033	Cast iron................	0.0162
Most ordinary sheet iron Transformer cores.....	0.0040 to 0.0045	Hardened cast steel.....	0.025

Assuming $y = 0.002$ the values of $\int H \, d\, I$ in column 3 of the following table have been obtained, by using Ewing's values of B in the previous table. The results compare very favorably with Ewing's values of $\int H \, d\, I$, which are reproduced in column 2.

$$\int H \, d\, I = 0.002\, B^{1.6}$$

B	$\int H\,d\,I$ Observed Ewing.	Computed.	Diff.	Diff. %.
1974	410	375	+ 35	+8
3830	1160	1081	+ 79	+7
5950	2190	2189	+ 1	0
7180	2940	2957	— 15	—2
8790	3990	4087	— 77	—2
10950	5560	5506	+ 54	+1
11480	6160	6265	—105	—2
11960	6590	6690	100	—2
13720	8690	8332	+258	+3
15560	10040	10190	—150	—2

We are, therefore, justified in assuming the loss of power in watts, per pound of iron, in steady cyclic change of n periods per second, where the maximum magnetic induction through the iron is B to be

$$w = 5.89 \times 10^{-6}\, n\, y\, B^{1.6} \ldots\ldots\ldots\ldots (395).$$

The armature of a two-pole dynamo makes 12 revolutions per second, in a field where the average number of magnetic lines per square centimetre is $B = 10{,}000$. The iron core weighs 20 lbs. If $y = 0.002$, and the density of iron be assumed 7.7 grammes per cubic centimetre, the loss in watts is

$$w = 5.89 \times 10^{-6} \times 12 \times 0.002 \times (10{,}000)^{1.6} \times 20 = 7.1.$$

The loss in watts per cubic centimetre of the iron is

$$w = \frac{n\, y\, B^{1.6}}{10^7} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (396)$$

$$= \frac{12 \times 0.002 \times (10000)^{1.6}}{10^7} = 0.006.$$

188. *Eddy currents in iron cores.*

Suppose the core to be of iron wire like that of a Gramme ring. Let a represent the radius of a wire, which is shown in section in Fig. 139. When *such a section of* any wire is at the *neutral points, the* induction through it will

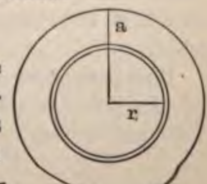


Fig. 139.

be a maximum. When the same section has revolved 90° it will stand edge-wise in the field and no lines will thread through it. A little later the lines begin to thread through the section in the opposite direction, and the number of such lines will reach a maximum at the other brush. The result is to develop currents circulating around the wire.

Let n = the number of revolutions per second in a two-pole machine, and let b = the number of lines threading through a wire section at any instant, the zero in t being when $b = 0$. Then will

$$b = B \sin 2\pi \frac{t}{T} = B \sin 2\pi n t \text{............} (397).$$

Consider a shell of radius r, concentric with the wire axis, as in Fig. 139. The induction through the section internal to the radius r will b $\pi r^2 b$. Then the $E.\ M.\ F.$ around the shell surface will be, in volts,

$$e = \frac{\pi r^2}{10^8} \frac{db}{dt}.$$

From (397)

$$\frac{db}{dt} = 2\pi n B \cos 2\pi n t$$

$$\therefore e = \frac{2\pi^2 n B r^2 \cos 2\pi n t}{10^8} \text{.........} (398).$$

The resistance of the cylindrical shell of radius r cm. and length 1 cm. to currents passing around it in the cylindrical conductor of radial thickness dr is in ohms

$$R = \frac{2\pi r}{c\, d r} \text{....................................} (399)$$

where $2\pi r$ is the length of the conductor, and $1 \times c\, dr$ is its section. The conductivity c of pure annealed iron wire at 0°C. is 1.02×10^5 with a decrease of 0.365 per cent. per degree C. For grey cast iron c at 0° C. is 9.469×10^3, the decrease per degree C. being 0.083 per cent.

The power in watts developed in a running centimetre of the wire is from (398) and (399)

$$\frac{w}{t} = \frac{e^2}{R} = \frac{2\pi^3 n^2 B^2 c \cos^2 2\pi n t}{10^{16}} \int_0^a r^3\, dr.$$

Remembering that the average value of cos^2 is $\frac{1}{2}$ and integrating we have

$$\frac{w}{2} = \frac{\pi^3 n^2 B^2 c a^4}{4 \times 10^{16}}$$

The loss in watts per cubic centimetre of the iron is, therefore, obtained by dividing this expression by the volume of the iron in which this power is applied, which is πa^2. We thus obtain

$$P = \frac{\pi^2 n^2 c B^2 a^2}{4 \times 10^{16}}.$$

In terms of the diameter $d = 2a$

$$P = \frac{\pi^2 n^2 B^2 c d^2}{16 \times 10^{16}} \qquad (400).$$

This represents the loss in energy per cubic centimetre per second, or during n periods. If we divide the expression by n, we shall have the energy lost in eddy currents during each cycle. This loss per cycle is, therefore, directly proportional to n, and may be reduced by slowing down the dynamo. In this respect this loss differs materially from that due to hysteresis. (Section 186.)

We may apply equation (400) to the armature discussed at the close of the last section. We have $B = 10{,}000$, $n = 12$. At a temperature of 20° C., $c = 0.94 \times 10^5$. We will assume the diameter of the iron wire to be 0.2 cm. The eddy current loss in watts per cubic centimetre is then by (400)

$$P = \frac{\pi^2 \times 144 \times 10^8 \times 0.94 \times 10^5 \times 0.04}{16 \times 10^{16}} = 0.0033.$$

We found the hysteresis loss in the same armature core to be 0.006 watts per cubic centimetre, and in a core weighing 20 lbs. the loss was 7.1 watts. The eddy current loss in the core would, therefore, be 3.9 watts.

If the iron is in the form of a thin plate, as in converters and drum armatures, the computation may be made as follows:—

Let a represent the thickness of the plate, which is

shown greatly enlarged with respect to the other dimension $m\,a$, in Fig. 140.

In the core of a drum armature this is a radial section of one of the plates of which the core is formed. In a converter, this figure may represent a section of one of the plates of the central core of the converter. In the armature, this section is continually changing its inclination with respect to the lines of force, being at right angles to them when in the plane of commutation, and standing edgewise when revolved a quarter of a revolution. This motion of the core varies the number of lines threading through the section. In the converter a similar periodic change in the magnetic lines is due to the periodic change in the current.

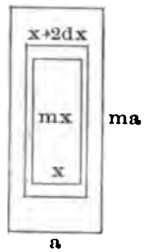


Fig. 140.

Within the section $a \times ma$, draw a similar figure whose dimensions are x and mx, and external to this figure draw another similar figure whose dimensions are $x + 2dx$ and $mx + 2mdx$.

As before we have for the number of lines threading through an unit area

$$b = B \sin 2\pi n t.$$

The elementary conductor here will be the rectangular shell around the central core $m\,x^2$. The length of the conductor is the distance around the rectangular core.

For each running centimetre measured at right angles to the plane of the figure, the section of the conductor will be dx on the long sides of the shell, and mdx at the ends of the core. As before

$$e = \frac{mx^2}{10^8}\,\frac{db}{dt}$$

$$\frac{db}{dt} = 2\pi n B \cos 2\pi n t$$

$$\therefore e = \frac{mx^2}{10^8}\, 2\pi n B \cos 2\pi n t.$$

The resistance of the elementary conductor is

$$R = \frac{2x}{cmdx} + \frac{2mx}{cdx}$$

$$= \frac{2x}{cdx}\left(\frac{1+m^2}{m}\right)$$

Hence

$$dw = \frac{e^2}{R} = \frac{4\,\pi^2\,n^2\,cm^3\,B^2\,cos^2\,2\,\pi\,nt}{2\,(m^2+1)\,10^{16}}\,x^3\,dx.$$

The power lost in a running centimetre is obtained by integrating in x between zero and a. For $cos^2\,2\,\pi\,nt$ we may also write its average value $\frac{1}{2}$.

Hence

$$w = \frac{\pi^2\,n^2\,c\,m^3\,B^2}{4\,(m^2+1)\,10^{16}}\,a^4.$$

This is the power lost in ma^2 cubic centimetres of the iron. The loss in watts per cubic centimetre is, therefore, obtained by dividing w by this volume, or,

$$P = \frac{\pi^2\,n^2\,c\,B^2\,a^2}{4 \times 10^{16}}\,\frac{m^2}{m^2+1} \ldots\ldots\ldots\ldots\ldots (401).$$

Here as in (400) the power lost is proportional to the square of the dimension which determines the degree of lamination, these dimensions being the diameter of the wire and the thickness of the plate.

The condition that the loss per cubic centimetre in round wire and in plates shall be the same is obtained by equating the values P in (400) and (401). We thus obtain,

$$d = 2a\,\sqrt{\frac{m^2}{m^2+1}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (402).$$

For comparatively small values of m this equation becomes

$$d = 2a$$

which is the condition of equal loss in ordinary drum and ring armatures. The thickness of the plate must be half the diameter of the equivalent wire. If $m = 1$, Eq. (402) given a similar comparison between round and square wire. If the loss per cubic centimetre is to be the same in two cases,

$$d = a\,\sqrt{2}$$

or the circumference of the round wire is the circumscribing circle to the section of the square wire. As will be readily seen, if the diameter of the round wire is *equal to the side of* the square wire, the loss per cubic

centimetre will be four times as great in the square wire as in the round.

It will be noticed that eddy current loss increases as the square of the frequency of alternation. Such losses may be greatly diminished by low frequency and by minute lamination. Eq. (401) does not apply to large wires, subject to rapid alternations, because the lines of induction do not penetrate deeply into iron cores under such conditions.

189. *Coefficient of self-induction in coils having iron cores.*

In section 120 the coefficient of self-induction of a coil of one winding was defined to be the number of lines due to the winding when it carries unit current. Since the strength of the field due to the current is proportional to the current, the number of lines due to a current i would be

$$N_1 = L_1 i.$$

If the coil have S windings it is evident that when unit current is passing there will be S times as many lines passing through the coil, and each line will link with S turns instead of with one. This is more exactly the case when the coil has an iron core which carries practically all of the lines due to each winding, and serves as a conducting channel for the magnetic field.

Equation (392) gives the number of lines carried by the core of an electro-magnet when the iron circuit is closed, as in a ring or a transformer. We have

$$N_1 = \frac{4\pi S i \mu A}{l}.$$

The number of lines carried by the core when the coil has unit current is then

$$N_1 = \frac{4\pi S \mu A}{l}.$$

But each of these lines links S times with the current to which they are due. The coefficient of self-induction of such a coil therefore becomes

$$L = N_1 S = \frac{4\pi S^2 \mu A}{l} \quad \text{..........(403).}$$

If an alternating $E.\ M.\ F.$ be applied to the terminals of this coil, the number of lines of induction will vary through a cycle as is represented by Eq. (397) if E varies according to the sine law. Since μ varies with the number of lines in the iron core, L will not be constant throughout the period, as it is when the magnetic circuit contains no iron. The value of L may, however, be experimentally determined by the method explained in section 170. The value of μ may then be computed from (403). The value of μ thus obtained will be subject to error due to eddy currents unless the iron is well laminated. The effect will be to make L and, therefore, μ too small.

The self-induction coefficient of the primary and the secondary coils of a transformer may be written as in Eq. (403)

$$L_1 = \frac{4\pi S_1^2 \mu A}{l}$$

$$L_2 = \frac{4\pi S_2^2 \mu A}{l}$$

$$\therefore \frac{L_1}{L_2} = \frac{S_1^2}{S_2^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (404)$$

where S_1 and S_2 are the number of windings in the two coils.

Remembering the definition given the coefficient of mutual induction in section 119, where coils of one winding were discussed, it is evident from what precedes that the number of lines due to the primary coil will be $\frac{4\pi S_1 \mu A}{l}$ when $i_1 = 1$ and that each of these lines link S_2 times with the secondary circuit. The coefficient of mutual induction, therefore, becomes

$$M = \frac{4\pi S_1 S_2 \mu A}{l} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (405).$$

For a given frequency this quantity determines the $E.\ M.\ F.$ induced in either coil when the other carries a unit periodic current. It is evident that

$$M^2 = L_1 L_2$$

or the coefficient of mutual induction where all of the lines due to either coil link with the other, is a mean proportional between the coefficients of self-induction.

CHAPTER XI.

UNITS.

190. *Current and Quantity.*

The units of current and quantity of electricity have been defined in the introduction. We have for the unit of quantity

$$Q = M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}$$

$$\overline{Q} = M^{\frac{1}{2}} L^{\frac{1}{2}}$$

$$I = M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}$$

$$\overline{I} = M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1}.$$

Magnetic Units.

191. *Magnet Pole.*

The equation which expresses the action between a magnet pole and a current is by section 117.

$$f = \frac{2 \pi i m}{r}$$

$$\therefore m = \frac{f r}{2 \pi i}.$$

The magnitude of the unit magnet pole is, therefore, directly proportional to that of unit force, and of unit length, and inversely as the magnitude of the unit current. It is, therefore, evident that we may measure the strength of the magnet pole in either electromagnetic or in electrostatic units. This will depend upon the unit in which i is measured. We have for the unit pole in electromagnetic measure.

$$\overline{M}_m = \frac{M L T^{-2} L}{M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1}} = M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}$$

This is the same result as is obtained from the equation for attraction between magnet poles. We have

$$f = \frac{m\, m'}{r^2}.$$

The equation for units will, therefore, be

$$F = \frac{\overline{M_m}^2}{L^2} = \frac{M\, L}{T^2}$$

$$\therefore\ \overline{M_m} = M^{\frac{1}{2}}\ L^{\frac{3}{2}}\ T^{-1}.$$

If we measure magnet pole in terms of the electrostatic unit of current we have

$$M_e = \frac{M\, L\, T^{-2}\, L}{M^{\frac{1}{2}}\ L^{\frac{3}{2}}\ T^{-2}} = M^{\frac{1}{2}}\ L^{\frac{1}{2}}.$$

192. *Magnetic Moment.*

Magnetic moment is magnetic pole or magnetic quantity into length. Its unit in electromagnetic measure will then be

$$\overline{MM_m} = M^{\frac{1}{2}}\ L^{\frac{5}{2}}\ T^{-1}.$$

In electrostatic measures its unit will be

$$MM = M^{\frac{1}{2}}\ L^{\frac{3}{2}}$$

193. *Intensity of magnetization.*

This was defined in sections 100 and 185 as magnetic moment per unit volume. In electromagnetic measure the unit for this quantity will be

$$\overline{I} = M^{\frac{1}{2}} L^{-\frac{1}{2}}\ T^{-1}.$$

The electrostatic unit is

$$I = M^{\frac{1}{2}}\ L^{-\frac{3}{2}}.$$

194. *Strength of magnetic field.*

The strength of a magnetic field is measured by the force acting upon a unit pole. If the pole is measured in electrostatic units the equation will then determine the field in such units, while if we measure the pole in electromagnetic units the field will then be measured in

such units also. For the electromagnetic unit of field strength we have

$$\overline{H} = \frac{F}{\overline{M}} = \frac{M\,L\,T^{-2}}{M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-1}} = M^{\frac{1}{2}}\,L^{-\frac{1}{2}}\,T^{-1}.$$

For the electrostatic unit

$$H = \frac{F}{M} = \frac{M\,L\,T^{-2}}{M^{\frac{1}{2}}\,L^{\frac{1}{2}}} = M^{\frac{1}{2}}\,L^{\frac{1}{2}}T^{-2}.$$

Magnetic field is also measured in electromagnetic units by Eq. (181)

$$H = \frac{2i}{r}.$$

Expressing the current unit in electromagnetic measure

$$\overline{H} = \frac{M^{\frac{1}{2}}\,L^{\frac{1}{2}}\,T^{-1}}{L} = M^{\frac{1}{2}}\,L^{-\frac{1}{2}}\,T^{-1}.$$

In electrostatic units

$$H = \frac{M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-2}}{L} = M^{\frac{1}{2}}\,L^{\frac{1}{2}}\,T^{-2}$$

which gives the same results as before.

195. *Magnetic potential.*

The work required to carry a unit pole around a current of strength i is $4\,\pi\,i$. This is a difference in magnetic potential. The units of magnetic potential will, therefore, be the same as for current, or

$$\overline{V} = M^{\frac{1}{2}}\,L^{\frac{1}{2}}\,T^{-1}$$

$$V = M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-2}.$$

196. *Magnetic flux or Number of magnetic lines.*

This quantity is defined by Green's equation

$$N = 4\,\pi\,m.$$

The number of lines of force $H\,A$ passing through a surface of area A which encloses m will depend on the permeability of the medium in which H is measured. If the permeability were perfect or $\mu = \infty$, H would be zero. (See section 98.) The number of lines of induction $B\,A = \mu\,H\,A$ is independent of the medium and is equal to

$4\,\pi\, m$. Hence in electromagnetic measure the unit of flux is the same as that for unit pole, or,

$$\overline{N} = M^{\frac{1}{2}}\, L^{\frac{3}{2}}\, T^{-1}.$$

And in electrostatic measure

$$N = M^{\frac{1}{2}}\, L^{\frac{1}{2}}.$$

197. *Magnetic resistance or Reluctance.*

This quantity is defined by an equation identical with Ohm's law. See Eq. (391.) The equation may be written

$$R_m = \frac{4\,\pi\, S\, i}{N}.$$

The electromagnetic unit of reluctance will then be

$$\overline{R}_m = \frac{M^{\frac{1}{2}}\;\; L^{\frac{1}{2}}\;\; T^{-1}}{M^{\frac{1}{2}}\;\; L^{\frac{3}{2}}\;\; T^{-1}} = L^{-1}.$$

The electrostatic unit is

$$R_m = \frac{M^{\frac{1}{2}}\; L^{\frac{3}{2}}\; T^{-2}}{M^{\frac{1}{2}}\; L^{\frac{1}{2}}} = L\; T^{-2}.$$

Permeance is the reciprocal of reluctance.

198. *Permeability.*

The reluctance of a magnetic conductor is by (391) equal to $\frac{l}{\mu A}$. By means of the results of section 197 we may, therefore, define the unit permeability. The electromagnetic unit is

$$\overline{unit\ \mu} = \frac{L}{L^{-1}\, L^2} = 1.$$

The electrostatic unit is

$$unit\ \mu = \frac{L}{L\, T^{-2}\, L^2} = L^{-2}\, T^2.$$

We may now return to section 195 and re-examine the equations which determine magnetic potential. In section 99 we wrote as the value for magnetic potential $\frac{m}{r}$.

If we by means of this equation determine the electro-magnetic unit for magnetic potential it is seen to be $\frac{M^{\frac{1}{2}}\ L^{\frac{3}{2}}\ T^{-1}}{L} = M^{\frac{1}{2}}\ L^{\frac{1}{2}}\ T^{-1}$ which agrees with the results of section 195. But the electrostatic unit is $\frac{M^{\frac{1}{2}}\ L^{\frac{1}{2}}}{L} = M^{\frac{1}{2}}\ L^{-\frac{1}{2}}$ which does not agree with the results of section 195. This discrepancy with others that might be pointed out has been the subject for general discussion. The solution of the difficulty seems to be that the potential due to a magnetic shell must depend on permeability in the same way that electrical potentials depend on specific inductive capacity. The more general expression for potential due to a magnetic shell is $\frac{m}{\mu r}$. Since the electromagnetic unit of μ is unity, potentials in that unit due to a shell are not affected by leaving μ out of consideration. Magnetic potential in electrostatic measure, therefore, becomes

$$V = \frac{M^{\frac{1}{2}}\ L^{\frac{1}{2}}}{L\ L^{-2}\ T^{2}} = M^{\frac{1}{2}}\ L^{\frac{3}{2}}\ T^{-2}.$$

This value for the unit is in harmony with the results of section 195.

199. *Magnetic induction.*

The product of permeability into magnetic force has been defined as the induction at a point, or $B = \mu\ H$. The electro-magnetic unit of magnetic induction is, therefore,

$$\overline{B} = M^{\frac{1}{2}}\ L^{-\frac{1}{2}}\ T^{-1}$$

which is identical with the unit for magnetic force, or strength of magnetic field. The electrostatic unit is

$$B = M^{\frac{1}{2}}\ L^{\frac{1}{2}}\ T^{-2} \times L^{-2}\ T^{2}$$
$$= M^{\frac{1}{2}}\ L^{-\frac{3}{2}}.$$

If we determine the integral $\int B\ d\ A$ over the face of a magnet, or over the area of any helix it will give the

total number of lines proceeding from the magnetic distribution, or from a magnetic shell equivalent to the helix. The unit in which this integral is measured is the same as that in which strength of pole or quantity of magnetism is measured. Magnetic induction is also magnetic flux per unit area.

200. *Magnetizing force.*

This quantity is determined by the number of units of current per centimetre in any helix. In section 114 this is given as $\frac{4\,\pi\,S\,i}{l} = 4\,\pi\,i_0$. The unit is the same as that for strength of field, one being proportional to the other. The electromagnetic unit is

$$\overline{H} = \frac{M^{\frac{1}{2}}\,L^{\frac{1}{2}}\,T^{-1}}{L} = M^{\frac{1}{2}}\,L^{-\frac{1}{2}}\,T^{-1}.$$

The electrostatic unit is

$$H = \frac{M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-2}}{L} = M^{\frac{1}{2}}\,L^{\frac{1}{2}}\,T^{-2}.$$

201. *Magnetomotive force.*

This is defined by an equation similar to Ohm's law. Eq. (391) or $F = N\,R$. The electromagnetic unit is

$$\begin{aligned}\overline{MMF} &= \overline{N}\,\overline{R} = \overline{B}\,L^2\,R \\ &= M^{\frac{1}{2}}\,L^{-\frac{1}{2}}\,T^{-1}\,L^2\,L^{-1} \\ &= M^{\frac{1}{2}}\,L^{\frac{1}{2}}\,T^{-1}.\end{aligned}$$

The electrostatic unit is

$$\begin{aligned}MMF &= M^{\frac{1}{2}}\,L^{-\frac{3}{2}}\,L^2\,L\,T^{-2} \\ &= M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-2}\end{aligned}$$

Equation (391) shows that this quantity is $4\,\pi\,S\,i$ and, therefore, the unit in which this quantity is measured is identical with that in which current is measured. *(Section 190.)*

202. *Reluctivity.*

This is the reciprocal of permeability. The units are, therefore,

$$\overline{Re} = 1$$

$$Re = L^2 T^{-2}.$$

203. *Susceptibility.*

This quantity is defined by equation (155). It is determined at any point in a mass of iron by the moment per unit volume, divided by the strength of field at the point. The units are, therefore,

$$\overline{K} = \frac{\overline{I}}{\overline{H}} = \frac{M^{\frac{1}{2}} L^{-\frac{3}{2}}}{M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-2}} = L^{-2} T^2$$

$$K = \frac{I}{H} = \frac{M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}}{M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}} = 1.$$

Electrical Units.

204. *Resistance.*

In section 131 we have shown that the electromagnetic unit of resistance is a velocity, and in sections 92–3 we have shown that the electrostatic unit is the reciprocal of a velocity. Hence

$$\overline{R} = L T^{-1}$$

$$R = L^{-1} T.$$

205. *Electrical potential or electro-motive force.*

This quantity is defined by Ohm's law. We have $E = R. I.$ For the electrostatic unit

$$E = R I = L^{-1} T M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2}$$

$$= M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1}.$$

For the electromagnetic unit

$$\overline{E} = \overline{R}\, \overline{I} = L T^{-1} M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1}$$

$$= M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2}.$$

206. *Strength of field.*

This has been defined as the force acting on unit quantity, or the force per unit quantity. In electrostatic units the unit will, therefore, be

$$\frac{F}{Q} = \frac{M\,L\,T^{-2}}{Q} = \frac{M\,L\,T^{-2}}{M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-1}}$$
$$= M^{\frac{1}{2}}\,L^{-\frac{1}{2}}\,T^{-1}.$$

The electromagnetic unit is

$$\frac{F}{\overline{Q}} = \frac{M\,L\,T^{-2}}{\overline{Q}} = \frac{M\,L\,T^{-2}}{M^{\frac{1}{2}}\,L^{\frac{1}{2}}} = M^{\frac{1}{2}}\,L^{\frac{1}{2}}\,T^{-2}.$$

207. *Flow of electrical lines or number of electrical lines.*

This quantity may be defined by Green's equation

$$N = 4\,\pi\,Q.$$

The electrostatic unit is

$$N = M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-1}.$$

The electromagnetic unit is

$$\overline{N} = M^{\frac{1}{2}}\,L^{\frac{1}{2}}.$$

208. *Diviance*, or resistance to *electrical lines of force.*

In section 83 we have seen that the number of lines of force proceeding from a quantity Q on a body having a potential E is

$$N = 4\,\pi\,Q = \frac{E}{R_d}$$

where R_d is the diviance of the surrounding space. The units for diviance are, therefore,

$$R_d = \frac{E}{N} = \frac{M^{\frac{1}{2}}\;L^{\frac{1}{2}}\;T^{-1}}{M^{\frac{1}{2}}\;L^{\frac{3}{2}}\;T^{-1}} = L^{-1}$$

$$\overline{R}_d = \frac{\overline{E}}{\overline{N}} = \frac{M^{\frac{1}{2}}\;L^{\frac{3}{2}}\;T^{-2}}{M^{\frac{1}{2}}\;L^{\frac{1}{2}}} = L\;T^{-2}.$$

Perviance is the reciprocal of diviance.

209. *Perviability or specific inductive capacity.*

The diviance of a tube of electrical induction is determined in the same manner that the reluctance of a tube of magnetic induction is determined. If K represents the specific-inductive capacity or perviability of the medium, the diviance of a tube of force of length l and section A is $R = \frac{l}{KA}$. The units in which K is measured are, therefore,

$$K = \frac{L}{L^2 \ L^{-1}} = 1.$$

In electrostatic measure, perviability is, therefore, independent of length, mass and time units. In electromagnetic measure

$$\overline{K} = \frac{L}{L^2 L \ T^{-2}} = L^{-2} \ T^2.$$

We may now return to section 205 and observe that the potential at a point distant r from a mass of electricity Q depends upon the perviability of the medium. This relation is entirely similar to the corresponding one in magnetic potential discussed in section 198. (See also section 85.) The potential is

$$E = \frac{Q}{KL} = \frac{M^{\frac{1}{2}} \ L^{\frac{3}{2}} \ T^{-1}}{L} = M^{\frac{1}{2}} \ L^{\frac{1}{2}} \ T^{-1}.$$

Since the unit in which K is measured is unity in the electrostatic system. In electromagnetic units

$$\overline{E} = \frac{\overline{Q}}{\overline{K} L} = \frac{M^{\frac{1}{2}} \ L^{\frac{1}{2}}}{L^{-2} \ T^{-2} \ L} = M^{\frac{1}{2}} \ L^{\frac{3}{2}} \ T^{-2}.$$

which agrees with the results of section 205.

210. *Electrical Induction.*

This represents the number of electrical lines per square centimetre. It is, therefore, obtained from section 207.

$$B' = \frac{N}{L^2} = \frac{M^{\frac{1}{2}} \ L^{\frac{3}{2}} \ T^{-1}}{L^2} = M^{\frac{1}{2}} \ L^{-\frac{1}{2}} \ T^{-1}.$$

$$\overline{B'} = \frac{\overline{N}}{L^2} = \frac{M^{\frac{1}{2}} \ L^{\frac{1}{2}}}{L^2} = M^{\frac{1}{2}} \ L^{-\frac{3}{2}}.$$

We may also derive the unit from sections 206 and 209

$$B' = \frac{F}{Q} K = M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}$$

since K is unity in electrostatic measure.

$$\overline{B'} = \frac{F}{\overline{Q}} \overline{K} = M^{\frac{1}{2}} \; L^{\frac{1}{2}} \; T^{-2} \; L^{-2} \; T^{2}$$

$$= M^{\frac{1}{2}} L^{-\frac{3}{2}}.$$

See section 199.

211. *Capacity.*

Capacity is defined by the equation $C = \frac{Q}{E}$.

The unit capacity in electrostatic measure is

$$C = \frac{Q}{E} = \frac{M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}}{M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1}} = L.$$

The electromagnetic unit is

$$\overline{C} = \frac{\overline{Q}}{\overline{E}} = \frac{M^{\frac{1}{2}} L^{\frac{1}{2}}}{M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2}} = L^{-1} \; T^{2}.$$

212. *Conductance.*

Conductance is the reciprocal of resistance. Its units are therefore,

$$G = L \, T^{-1}$$

$$\overline{G} = L^{-1} T.$$

213. *Conductivity, or specific-conductance.*

The resistance of any conductor is

$$R = \frac{l}{\gamma A}$$

where γ is the conductivity and $\frac{1}{\gamma}$ is the resistivity. Hence $\gamma = \frac{l}{A\,R}$ and the units are

$$\gamma = \frac{L}{L^{2} L^{-1} T} = \frac{1}{T}$$

$$\bar{\gamma} = \frac{L}{L^{2} L \, T^{-1}} = L^{-2} \; T$$

The units of resistivity are the reciprocals of these values or

$$\rho = T$$

$$\overline{\rho} = L^{2} T^{-1}$$

214. *Coefficients of self-induction and of mutual induction.*

By Equation (403) it will be seen that the electrostatic unit in which the coefficient of self-induction is measured is

$$L = \frac{\mu L^{2}}{L} = \frac{L^{-2} T^{2} L^{2}}{L} = L^{-1} T^{2}.$$

The electromagnetic unit is

$$\overline{L} = \frac{\overline{\mu} L^{2}}{L} = L.$$

The coefficient of mutual induction is measured in the same units as is shown by Eq. (405).

215. *Impedance.*

The quantity called impedance is $\sqrt{R^2 + 4\pi^2 \frac{L^2}{T^2}}$ Eq. (305). Since the coefficient of self-induction L is a length and T is a time, it is evident that the electromagnetic unit for $\frac{L}{T}$ is a velocity, which is also the unit in which R is measured. The quantity $\sqrt{R^2 + 4\pi^2 \frac{L^2}{T^2}}$ therefore, is measured in velocity or in resistance units. The electrostatic unit of impedence is, therefore, the reciprocal of a velocity. The term $\frac{T}{2\pi C}$ of equation 322 also has the same unit as resistance.

216. *Power and energy in electrical units.*

The measure of power in electrical units is $Ei = Ri^2 = \frac{E^2}{R}$. In electrostatic units the unit in which this quantity is measured is

$$EI = M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} = M L^2 T^{-3}$$

$$RI^2 = L^{-1} T M L^3 T^{-4} = M L^{2} T^{-3}$$

$$\frac{E^2}{R} = \frac{M L T^{-2}}{L^{-1} T} = M L^{2} T^{-3}$$

In electromagnetic measure

$$\overline{E}\,\overline{I} = M^{\frac{1}{2}}\ L^{\frac{3}{2}}\ T^{-2} M^{\frac{1}{2}}\ L^{\frac{1}{2}}\ T^{-1}$$
$$= M L^2\ T^{-3}$$

It is evident that the units $\overline{R}\ \ \overline{I}^2$ and $\dfrac{\overline{E}^2}{\overline{R}}$ are measured in the same unit. The words electrical power or electrical energy, are correct only in the sense that they indicate that electrical machinery is being used. In the same sense we may also say steam power and steam energy.

The $C.\ G.\ S.$ unit of power, both in the electromagnetic system and in the electrostatic system is the same. It is the erg per second. The unit of work is

$$E\,I\,T = M\,L^2\,T^{-2} = \overline{E}\ \ I\ \ T.$$

217. *Tabulation of units.*

We may now collect the values of the units in a table. It will be observed that the electromagnetic unit in which any magnetic quantity is measured is identical with the electrostatic unit in which any similar electrical quantity is measured. For example, the electromagnetic unit of magnetic potential and the electrostatic unit of electrical potential are identical, being $M^{\frac{1}{2}}\ L^{\frac{1}{2}}\ T^{-1}$. The electrostatic unit of magnetic potential and the electromagnetic unit of electrical potential are $M^{\frac{1}{2}}\ L^{\frac{3}{2}}\,T^{-2}$. The same relations hold for permeability and perviability or specific inductive capacity, for reluctance and diviance for magnetomotive force and electromotive force, etc. The arrangement in the table is such as to give analogous quantities similar positions, so far as this is feasible.

The final column in the table is the ratio of electrostatic to electromagnetic units. It is at once seen that this ratio in case of every electrical and every magnetic quantity is a function of $L\ T^{-1}$ only, which is of the character of a velocity. This velocity is a quantity independent of any particular systems of measurement. It depends for its magnitude on the machinery concerned in the repulsions (1) of equal masses of electricity for each other, (2) of equal masses of magnetism which act on each

If there are n cycles per
in ergs per second per poun
tiplying by n.

Since 10^7 ergs per secon
in watts per pound of iron

$$w = \frac{453.}{7.7 \times}$$

$$= 5.89$$

Steinmetz has found
which the energy lost
cycle may be expressed
B. He finds for any cy

$$\int H$$

where y is a constant
the following values f

Material.
Very soft iron wire...
Very thin soft sheet ir
Thin good sheet iron..
Thick sheet iron......
Most ordinary sheet ir
Transformer cores...

Assuming $y =$
3 of the follow
Ewing's values
compare very f
which are repr

ELECTRICAL.

Physical Quantity.	Units. Electro-static.	Units. Electro-magnetic.	Ratio $\frac{E.S.}{E.M.}$
Quantity	$M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-1}$	$M^{\frac{1}{2}}\,L^{\frac{1}{2}}$	$L\,T^{-1}$
Intensity of field	$M^{\frac{1}{2}}\,L^{-\frac{1}{2}}\,T^{-1}$	$M^{\frac{1}{2}}\,L^{\frac{1}{2}}\,T^{-2}$	$L^{-1}\,T$
Potential	$M^{\frac{1}{2}}\,L^{\frac{1}{2}}\,T^{-1}$	$M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-2}$	$L^{-1}\,T$
Induction	$M^{\frac{1}{2}}\,L^{-\frac{1}{2}}\,T^{-1}$	$M^{\frac{1}{2}}\,L^{-\frac{3}{2}}$	$L\,T^{-1}$
Flux, Flow of lines	$M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-1}$	$M^{\frac{1}{2}}\,L^{\frac{1}{2}}$	$L\,T^{-1}$
Diviance	L^{-1}	$L\,T^{-}$	$L^{-2}\,T^{2}$
Perviance	L	$L^{-1}\,T^{2}$	$L^{2}\,T^{-2}$
Perviability, Sp. ind. capacity	1	$L^{-2}\,T^{2}$	$L^{2}\,T^{-2}$
Diviability	1	$L^{2}\,T^{-2}$	$L^{-2}\,T^{2}$
Electromotive force	$M^{\frac{1}{2}}\,L^{\frac{1}{2}}\,T^{-1}$	$M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-2}$	$L^{-1}\,T$
Resistance	$L^{-1}\,T$	$L\,T^{-1}$	$L^{-2}\,T^{2}$
Conductance	$L\,T^{-1}$	$L^{-1}\,T$	$L^{2}\,T^{-2}$
Current	$M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-2}$	$M^{\frac{1}{2}}\,L^{\frac{1}{2}}\,T^{-1}$	$L\,T^{-1}$
Surface density	$M^{\frac{1}{2}}\,L^{-\frac{1}{2}}\,T^{-1}$	$M^{\frac{1}{2}}\,L^{-\frac{3}{2}}$	$L\,T^{-1}$
Capacity	L	$L^{-1}\,T^{2}$	$L^{2}\,T^{-2}$
Resistivity	T	$L^{2}\,T^{-1}$	$L^{-2}\,T^{2}$
Conductivity	T^{-1}	$L^{-2}\,T^{1}$	$L^{2}\,T^{-2}$
Coeff. of Self and of Mutual Induction	$L^{-1}\,T^{2}$	L	$L^{-2}\,T^{2}$
Impedance	$L^{-1}\,T$	$L\,T^{-1}$	$L^{-2}\,T^{2}$
Power	$M\,L^{2}\,T^{-3}$	$M\,L^{2}\,T^{-3}$	1
Energy	$M\,L^{2}\,T^{-2}$	$M\,L^{2}\,T^{-2}$	1

An inspection of the preceding table shows that the ratio of electrostatic to electromagnetic units for quantity, potential, current, resistance and capacity, is as follows:—

$$\frac{Q}{\overline{Q}} = \frac{\overline{V}}{V} = \frac{I}{\overline{I}} = 3 \times 10^{10}$$

$$\frac{\overline{R}}{R} = \frac{C}{C'} = 9 \times 10^{20}$$

or in *C. G. S.* units

1 electromagnetic unit of		quantity $= 3 \times 10^{10}$	electrostatic units
1	"	current $= 3 \times 10^{10}$	"
3×10^{10}	"	potential $= 1$	"
1	"	capacity $= 9 \times 10^{20}$	"
9×10^{20}	"	resistance $= 1$	"

The relations of other quantities may be easily made out in a similar way.

The electrostatic capacity of a sphere of metal having a radius of one-half metre is 50 *C. G. S.* units. Its capacity in electromagnetic *C. G. S.* units will, therefore, be $\frac{50}{9 \times 10^{20}}$.

218. *Practical units.*

Magnetic values are usually measured in *C. G. S.* electromagnetic units. For example, the horizontal component of the earth's magnetic field is about 0.2 in *C. G. S.* electro-magnetic units. The magnetic field at a distance of 10 centimetres from a wire carrying 12 *C. G. S.* electromagnetic units of current is $\frac{2 \times 12}{10} = 2.4$. The strength of field in the air-gap of a dynamo may be 5,000 to 10,000. This is often expressed in other words by saying that in the air-gap of a dynamo there may be 5,000 to 10,000 *C. G. S.* magnetic lines to the square centimetre, or the induction is 5,000 to 10,000 *C. G. S.* lines. Electrostatic measurements of electrical quantities are also expressed in *C. G. S.* units. The electrostatic capacity of a sphere of 50 centimetres radius is 50. But in electromagnetic units its capacity is

$$\frac{50}{9 \times 10^{20}} = 5.55 \times 10^{-20}.$$

Capacities such as are commonly dealt with are, therefore, seen to be represented by very small fractions, when expressed in electromagnetic *C. G. S.* units. Since condensers are used in determinations made in electromagnetic measure it is necessary to measure capacities in such units. Likewise electrical quantities are expressed in very small fractions, while resistances and electromotive forces such as are commonly used when expressed in *C. G. S.* electromagnetic units are represented by very large numbers. The *E. M. F.* of a gravity cell is 107,000,000. Such numbers could not be tolerated in engineering practice, and, therefore, a set of practical units has been devised, which have a simple relation to the *C. G. S.* units. The change is made to affect the units of all the quantities to be measured, in a way that was thought to be on the whole most desirable. This was done by operating on the fundamental units of length and mass.

The practical unit of length is 10^9 centimetres. This is about equal to a quadrant of a meridian circle of the earth.

The unit of mass on which the practical system is based is 10^{-11} gramme. This mass bears about the same relation to the gramme, that one-third of a square inch bears to a square mile.

The unit of time in the practical system remains the second, as in the *C. G. S.* system.

The Volt. From the table we find that the *C. G. S.* electromagnetic unit of potential is $M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2}$. The practical unit of potential will, therefore, be

$$(10^{-11} M)^{\frac{1}{2}} (10^9 L)^{\frac{3}{2}} T^{-2} = M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} \times 10^{-\frac{11}{2}} \times 10^{\frac{27}{2}}$$
$$= M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} \times 10^8.$$

The practical unit is, therefore, 10^8 times as large as the *C. G. S.* unit. It is called the volt. The *E. M. F.* of a gravity cell is 107,000,000 *C. G. S.* units or 1.07 volts.

The Ohm. The *C. G. S.* unit of resistance in electromagnetic measure is $L\,T^{-1}$. The practical unit is, there-

fore, $L\ T^{-1} \times 10^9$, which is 10^9 times as great as the *C. G. S.* unit. This unit is called the ohm.

The Ampere. The *C. G. S.* unit of current is $M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1}$. The practical unit is then $M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} \times 10^{-\frac{11}{2}} \times 10^{\frac{9}{2}} = M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} \times 10^{-1}$.

The practical unit of current is, therefore, one-tenth of a *C. G. S.* unit. This unit is the ampere.

The Culomb. The *C. G. S.* unit of quantity is $M^{\frac{1}{2}} L^{\frac{1}{2}}$. The practical unit is $M^{\frac{1}{2}} L^{\frac{1}{2}} \times 10^{-\frac{11}{2}} \times 10^{\frac{9}{2}}$ $= M^{\frac{1}{2}} L^{\frac{1}{2}} \times 10^{-1}$, which is one-tenth of a *C. G. S.* unit. This unit is the culomb.

The Farad. The *C. G. S.* unit of capacity is $L^{-1}\ T^2$. The practical unit is $L^{-1}\ T^2 \times 10^{-9}$. The practical unit is, therefore, $\frac{1}{1,000,000,000}$ of a *C. G. S.* unit. This unit is called the farad.

The Henry. The *C. G. S.* unit in which the coefficients of mutual and self-induction are measured is L. The practical unit is 10^9 times as large or $L \times 10^9$. This unit is called the henry.

The *C. G. S.* unit of conductance is $L^{-1}\ T$. The practical unit is $L^{-1}\ T \times 10^{-9}$ which is $\frac{1}{10^9}$ as great as the *C. G. S.* unit. The practical unit is sometimes called the mho, this being the reversed spelling of ohm. A wire having a resistance of one ohm has a conductance of one mho. A wire having a resistance of 10 ohms has a conductance of $\frac{1}{10}$ mho.

The *C. G. S* unit for strength of magnetic field, or number of lines of force per square centimetre, is $M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}$. The practical unit is, therefore,

$$M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} \times 10^{-\frac{11}{2}} \times 10^{-\frac{9}{2}} = M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} \times 10^{-10}.$$

This unit is, therefore, $\frac{1}{10^{10}}$ as great as the *C. G. S.*

unit.* The unit of length in the practical system being 10^9 and the unit area 10^{18} times the *C. G. S.* units, the number of *C. G. S.* lines per unit area in the practical system becomes

$$M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} \times 10^{-10} \times L^2 \times 10^{18} = M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} \times 10^8.$$

In the practical system there will, therefore, be in a unit field 10^8 *C. G. S.* lines on an area 10^{18} square centimetres. This area if in the form of a square would have each side equal to an earth-quadrant in length. The number of square centimetres for each *C. G. S.* line becomes 10^{10}. This is an area 10,000 metres square.

A wire having a length equal to an earth quadrant, and sweeping over a distance of an earth quadrant each second where the strength of field at right angles to the plane swept over is one *C. G. S.* line to 10^{10} square centimetres, would develop an *E. M. F.* of one volt. The wire would cut 10^8 *C. G. S.* lines per second, or in practical units it would cut one line. The line of force in the practical system, is 10^8 *C. G. S.* lines. Any wire, long, or short, which cuts one practical line per second will develop an *E. M. F.* of one volt.† In a field having a strength of 10,000 *C. G. S.* lines per square centimetre, a wire 30 centimetres long moving with a velocity of 333 centimetres or 3.33 metres per second, would cut 10^8 *C. G. S.* lines per second.

The *C. G. S.* unit of power is the erg per second. Its unit is symbolically represented by $M L^2 T^{-3}$. The practical unit of power is, therefore, $M L^2 T^{-3} \times 10^{-11} \times 10^{18} = M L^2 T^{-3} \times 10^7$. This unit is, therefore, 10^7 ergs per second. It is called the watt. The *C. G. S.* unit of energy or the erg, or the dyne-centimetre, is $M L^2 T^{-2}$. The practical unit of energy is, therefore,

* The American Institute of Electrical Engineers recommends that the *C. G. S.* unit of magnetic field, or one *C. G. S.* line to the square centimetre, be called a *gauss*.

† The American Institute of Electrical Engineers, calls one *C. G. S.* magnetic line, a *weber*. The weber is therefore the *C. G. S.* unit of flux. The Institute also calls the *C. G. S.* unit of reluctance, the *œrsted*, and the *C. G. S.* unit of magneto-motive force the *gilbert*. *This unit is due to* $\frac{10}{4\pi}$ ampere turns (see Section 201).

$M L^2 T^{-2} \times 10^7$. This unit is, therefore, 10^7 ergs. It is called the joule. One joule = one watt-second, and the watt is a rate of work of one joule per second. The joule is 0.1019 kilogrammemetres, or about one-tenth kgr. metre. The horse-power = 746 watts, and the horse-power-hour = 2,685,600 joules, or 2.69 mega-joules. The Kilowatt is 1,000 watts. The prefix micro indicates one millienth of the unit. A micro-farad is one millionth of a farad. The prefix milli means one thousandth. A coil having a self-induction coefficient of 0.0019 henry this value may be expressed as 1.9 milli-henry.

Relation between Practical units and C. G. S. Electrostatic units.

Quantity. The *culomb* is the amount of electricity furnished each second to a lamp requiring a current of one ampere. It is the ampere-second. It is equal to $\frac{1}{10}$ *C. G. S.* electromagnetic unit. Hence by sections 217 and 218

$$1 \text{ culomb} = \frac{1}{10}\ C.\ G.\ S. \text{ electromagnetic unit}$$

$$= \frac{1}{10} \times 3 \times 10^{10} = 3 \times 10^9\ E.\ S. \text{ units}$$

$$\text{and } 1\ E.\ S. \text{ unit} = \frac{1}{3 \times 10^9} \text{ culomb.}$$

Current.

$$\begin{aligned} 1 \text{ ampere} &= \tfrac{1}{10}\ C.\ G.\ S.\ E.\ M. \text{ unit} \\ &= \tfrac{1}{10} \times 3 \times 10^{10}\ C.\ G.\ S.\ E.\ S. \\ &= 3 \times 10^9 \quad \text{"} \end{aligned}$$

$$\text{and } 1\ C.\ G.\ S.\ E.\ S. \text{ unit} = \frac{1}{3 \times 10^9} \text{ ampere.}$$

Potential.

$$1 \text{ volt} = 10^8\ C.\ G.\ S.\ E.\ M. \text{ units.}$$

$$= \frac{10^8}{3 \times 10^{10}} = \frac{1}{300}\ C.\ G.\ S.\ E.\ S. \text{ unit}$$

$$\text{and } 1\ C.\ G.\ S.\ E.\ S. \text{ unit} = 300 \text{ volts.}$$

* The units recommended by the American Institute of Electrical Engineers, correspond to the dyne and the erg, instead of to the ohm, volt, ampere, etc. The unit of force corresponding to these latter units would be $M L T^{-2} \times 10^{-11} \times 10^9 = M L T^{-2} \times 10^{-2}$. It would, therefore, be 0.01 dyne. This unit is not in use. The unit of work would, t[illegible] 10^7 ergs, as already stated. This unit is [illegible] joule. [illegible] done in exerting 0.01 dyne over a practical [illegible] of len[illegible] [illegible]tres.

Resistance.

$$1\,\text{ohm} = 10^9\ C.\ G.\ S.\ E.\ M.\ \text{unit.}$$

$$= \frac{10^9}{(3 \times 10^{10})^2} = \frac{1}{9 \times 10^{11}}\ C.\ G.\ S.\ E.\ S.$$

$$\therefore 1\ C.\ G.\ S.\ E.\ S.\ \text{unit} = 9 \times 10^{11}\ \text{ohms.}$$

$$= 900{,}000\ \text{megohms.}$$

Capacity.

$$1\ \text{farad} = \frac{1}{10^9}\ C.\ G.\ S.\ E.\ M.$$

$$= \frac{(3 \times 10^{10})^2}{10^9}\ C.\ G.\ S.\ E.\ S.$$

$$= 9 \times 10^{11}\ C.\ G.\ S.\ E.\ S.$$

$$\therefore 1\ C.\ G.\ S.\ E.\ S.\ \text{unit} = \frac{1}{9 \times 10^{11}}\ \text{farad.}$$

$$= \frac{1}{900{,}000}\ \text{micro-farad.}$$

Self-induction.

$$1\ \text{henry} = 10^9\ C.\ G.\ S.\ E.\ M.\ \text{units,}$$

$$= \frac{10^9}{(3 \times 10^{10})^2} = \frac{1}{9 \times 10^{11}}\ C.\ G.\ S.\ E.\ S.$$

$$\therefore 1\ C.\ G.\ S.\ E.\ S.\ \text{unit} = 9 \times 10^{11}\ \text{henrys.}$$

CHAPTER XII.

PROBLEMS.

219. *Attraction between spheres.*

Two conducting spheres have radii of ten centimetres, and the distance between their centres is 300 centimetres. They are at the same potential v volts. Find v if the repulsion of one on the other is equal to the weight of one milligramme.

The potential of each sphere is v volts $= \frac{v}{300}$ *C. G. S. E. S.* (by the last section).

The electrostatic capacity of each sphere being 10, the electrostatic quantity on each is $Q = 10 \frac{v}{300} = \frac{v}{30}$. The repulsion between the two spheres is to be 0.981 dyne. Hence by the equation for attraction Eq. (6)

$$0.981 = \frac{v^2}{900 \times 90000}$$

$$\therefore v = 8910 \text{ volts.}$$

The charge on each sphere is 297 *E. S.* units, = 0.000,000,099 culomb. (See last section.)

220. *Astronomical units of mass.*

Two solid spheres of copper, the density of which is 8.788 grammes per cubic centimetre, have radii 4.743 centimetres. The mass of each sphere will be 3928 grammes. This is the astronomical unit of mass. (Sections 10 and 21.) These spheres have a common potential v volts, which is so adjusted that their gravitation attraction is balanced by their electrical repulsion. Find v. As in the previous section, the electrostatic potential of [illegible] sphere is $\frac{v}{300}$ electrostatic units. The quantity [illegible] ere is $C\ V = r\ V = 4.743 \frac{v}{300}$.

The attraction of each charge on the other, D being the distance being centres, is

$$A = \left(\frac{4.743\,v}{300\,D}\right)^2.$$

The attraction between the two units of mass at unit distance being one dyne, we have for their attraction at a distance D

$$A = \frac{1}{D^2} \text{ hence}$$

$$\frac{1}{D^2} = \frac{(0.01581\ v)^2}{D^2}$$

$$\therefore v = 63.2 \text{ volts.}$$

The quantity on each sphere is then

$$Q = 63.2\ \frac{4.743}{300} = 1\ C.\ G.\ S. \text{ unit}$$

$$= \frac{1}{3 \times 10^9} = 0.000,000,000,33 \text{ coulomb.}$$

221. *Interaction of earth and moon.*

Suppose the earth and moon to be electrically charged with charges equivalent to their masses. What would be the potentials on the two bodies?

For each 3928 grammes of matter in either sphere, we must place on its surface one electrostatic unit. The mass of the earth in astronomical units being $\frac{6.14 \times 10^{27}}{3928}$, we must place this number of electrostatic units upon it. The radius of the earth being 6.37×10^8 centimetres, its potential will be in electrostatic units,

$$V = \frac{6.14 \times 10^{27}}{3928 \times 6.37 \times 10^8} = 2.45 \times 10^{15}$$

which by section 218 is 7.35×10^{17} volts.

The amount of electricity on each square centimetre of the earth's surface will be

$$\sigma = \frac{6.14 \times 10^{27}}{3928 \times 4\,\pi\,(6.37 \times 10^8)^2}$$
$$= 1.532 \times 10^5\ C.\ G.\ S.$$

By section 90 the force with which each square c metre would be impelled outward by the apparent

repulsion of the charged surface and an unit of its area is

$$\frac{dp}{dS} = 2\,\pi\,\sigma^2 = 2\,\pi\,(1.532 \times 10^5)^2$$
$$= 1.476 \times 10^{11} \text{ dynes.}$$

Dividing by 981 and by 1,000 the force in kilogrammes per square centimetre is 150,250. This is 107 tons of 2,000 lbs. to the square inch. The steel used in engineering structures will break at one-half of this force, and the best wire ropes will part at 124 tons to the square inch. The mass of the moon is $\frac{1}{80}$ that of the earth, and its radius is 1.74×10^8 centimetres. Using this data the computation for the moon may be readily made.

222. *Capacity of the earth.*

The radius of the earth being 6.37×10^8 centimetres, this is also its capacity in electrostatic units. By section 218 its capacity in farads is $\frac{6.37 \times 10^8}{9 \times 10^{11}}$ $= 0.0007077$ farads, $= 707.7$ micro-farads.

If two equal spheres having the size of the earth be charged to potentials of $+25$ and -25 volts, the difference in potential between them would be 50 volts. Neglecting their mutual action on each other, the capacity of each would be 0.000708 farad, and the charge on each would be $Q = C\,V = 25 \times 0{,}000708 = 0.0177$ culombs. Suppose these spheres to be connected with each other by a wire of no resistance having a 50 volt, one ampere lamp in circuit. A current of one ampere would be established in the circuit. To maintain the potentials of the spheres constant and thus keep the current in the lamp constant, the radii of the two spheres must diminish at a uniform rate. (See section 93.) Since $\frac{Q}{r} = V$ in each case, V being constant, and r being either of the radii at any instant, we have

$$dQ = V\,dr = i\,dt.$$

Integrating in t during the interval required for the radii to diminish from 6.37×10^8 to zero we have

$$t = \frac{V}{i}\,r.$$

By section 218, $V = 50$ volts $= \frac{50}{300}$ electrostatic units, and $i =$ one ampere or 3×10^9 *E. S.* units. The radii must, therefore, diminish to zero and the flow will cease in a time

$$t = \frac{50}{300} \frac{6.37 \times 10^8}{3 \times 10^9} = 0.0354 \text{ second.}$$

The velocity with which the radius must shorten is $\frac{r}{t} = \frac{i}{V} = 1.8 \times 10^{10}$ centimetres per second.

When the earth is used as a return circuit in an electric street car service, we are, therefore, not to think of the earth as a great electrical reservoir from which electricity is pumped by the dynamo, and into which it is again discharged. The operation resembles a pumping service in which water is pumped from the ocean and discharged into the ocean, where the amount of water flowing through a section of the pipe during one second is vastly greater than the pipe and ocean bed could hold at any instant. This would require that the same water shall pass through the circuit many times a second.

223. *Capacity*. We may determine what must be the area of two conducting sheets separated by a sheet of gutta percha one millimetre in thickness, if the capacity of the condenser is to be one farad or 9×10^{11} electrostatic units. By section 75 the capacity of a plate condenser of area S and with a distance t between the plates is $\frac{S}{4\pi t}$. This is when the plates are separated by air. The insulating medium being gutta percha having a specific inductive capacity or perviability 2.48, the capacity becomes $\frac{2.48\, S}{4\, \pi\, t}$.

Hence

$$9 \times 10^{11} = \frac{2.48\, S}{4\, \pi\, t}$$

or

$$S = \frac{4\, \pi \times 9 \times 10^{11}}{2.48} t = 4.56 \times 10^{12} t.$$

The area of the plates must be directly proportional

to the distance between them, the area being, however, large compared with the distance t. If $t =$ one millimetre $= 0.1$ cm.

$$S = 4.56 \times 10^{11} \text{ square centimetres.}$$

This would require square conducting sheets whose sides are 675,300 centimetres or 6753 metres, or six and three-fourths kilometres. Arranged as two sheets, only one side of the plates is available for condenser action. If cut into sheets of one square metre and interleaved with each other and with the insulations, both sides become useful. Each plate of the two-sheet condenser would yield 4.56×10^7 sheets having an area of one square metre. If the plates were a millimetre in thickness, the farad condenser would consist of a pile of 4.56×10^7 metal sheets and an equal number of insulating sheets. This makes in all 9.12×10^7 sheets, each of which is 0.1 cm in thickness. The thickness of the pile would be 9.12×10^6 centimetres $= 91.2$ kilometres or 56.6 miles. The capacity of two of the plates one metre square is 2.192×10^{-8} farad or 0.0219 micro-farad. As plates of the same size are added, the increase in capacity per plate is 2.192×10^{-8} farad. A condenser having a capacity of one farad, charged to a potential of one volt would have a charge of one culomb.

224. *Attracted disc Electrometer.*

By section 88 the attraction between the disc of the electrometer, and the attracting disc is

$$A = \frac{S}{4 \pi t^2} (V_1 - V_2)^2$$

where S is the area of the attracted disc and t the distance between the attracted and the attracting plates.

Suppose we wish an electrometer to measure a potential of 500 volts. This is $\frac{5}{3}$ electrostatic unit. Then the dimensions must satisfy the equation

$$S = \frac{36 \pi}{25} A t^2 = 4.52 A t^2.$$

If the pull on the attracted plate is to be equal to the weight of one decigramme (where $g = 980$), or 98 dynes, then

$$S = 442.96 t^2.$$

If t be made 0.5 centimetre, $S = 110.74$ square centimetres. This area should be the mean of the areas of the attracted plate, and of the opening in the guard-ring, into which the attracted plate should fit as closely as is consistent with freedom of motion.

225. *Sphere surrounded by a spherical shell.*

A sphere having a radius r_0, is surrounded by a concentric spherical shell having a perviability μ. The radii of the shell are r_1 and r_2. The sphere is connected with one brush of a dynamo, the other being grounded. The potential of the sphere is 2700 volts above that of the ground. This is 9 electrostatic units. Let $r_0 = 20$ centimetres. Then the quantity on the sphere is 180 *E. S.* units or $\dfrac{180}{3 \times 10^9} = 6 \times 10^{-8}$ culombs when the sphere is in free space. When surrounded by the shell of perviability μ, the fictive layer on each surface is by (106)

$$Q_1 = Q\left(1 - \frac{1}{\mu}\right)$$

The potential of the sphere is then

$$V = \frac{Q}{r_0} - \frac{Q_1}{r_1} + \frac{Q_1}{r_2} = \frac{Q}{r_0} - Q\left(1 - \frac{1}{\mu}\right)\frac{r_2 - r_1}{r_2\, r_1}.$$

If $r_1 = 21$; $r_2 = 22$, and $\mu = 2.5$ then by the conditions already imposed

$$9 = \frac{Q}{20} - Q \times 0.4\, \frac{1}{21 \times 22}$$

$$\therefore Q = 183.3 \text{ electrostatic units.}$$

$$Q' = 73.3 \quad \text{"} \quad \text{"} \quad \text{"}$$

The capacity of the sphere when surrounded by the shell is

$$C = \frac{183.3}{9} = 20.37 \ E.\ S. \text{ units.}$$

or $\dfrac{20.37}{9 \times 10^5}$ micro-farads. When in free space the capacity of the sphere is 20 *E. S.* units, or $\dfrac{20}{9 \times 10^5}$ micro-farads.

(Section 218). If the shell be supposed a conductor, then $\mu = \infty$ and $Q' = Q$ and

$$V = Q\left(\frac{1}{r_0} - \frac{1}{r_1} + \frac{1}{r_2}\right)$$

or

$$9 = Q\left(\frac{1}{20} - \frac{1}{21} + \frac{1}{22}\right)$$

$$\therefore Q = 188.2$$

$$\text{and } C = \frac{188.2}{9} = 20.9.$$

If the shell be grounded its potential will be zero. The potential of the sphere will then be

$$V = Q\left(\frac{1}{r_0} - \frac{1}{r_1}\right)$$

since there will be no charge on the outer surface of the shell.

The quantity on the sphere will be determined by the equation

$$9 = Q\left(\frac{1}{20} - \frac{1}{21}\right) = Q\ \frac{1}{20 \times 21}$$

$$\therefore Q = 3780.$$

The capacity of the sphere is now

$$C = \frac{3780}{9} = 420\ E.\ S.\ \text{units}$$

$$\text{or } \frac{420}{9 \times 10^5} \text{ micro-farad.}$$

The enormous increase in capacity in this case is not due to the fact that the shell is grounded, but to the fact that one pole of the dynamo is connected with the sphere, and the other with the shell, and the device as a whole is being used as a condenser. In the former cases when the shell was insulated and contained equal charges of unlike sign, it affected the capacity of the sphere as any neighboring body would affect it.

If neither side of the dynamo were grounded, the potential of the sphere would be + 1350 volts and that of the shell — 1350. In electrostatic measure these potentials would be 4.5. The charge on the sphere must necessarily attract an equal charge of opposite sign to the inner surface of the conducting shell. Any free

charge on the shell would necessarily result in producing constant potential at all internal points and could not, therefore, affect the difference in potential between sphere and shell.

The difference in potential between sphere and shell would, therefore, be 9 *E. S.* units as before. The actual potentials of sphere and shell would depend upon the amount of free charge on the shell. If there were none the potential of the shell would be zero and the condition would be the same as that just discussed when the shell was grounded. If the dynamo were perfectly insulated, it would take part of the free charge of the shell, so that the free charge on the shell would always be less than the bound charges $+Q$ and $-Q$. Call it Q'. There could be no free charge on the sphere, since it is completely surrounded by the shell.

The potential of the sphere would then be

$$V_0 = \frac{Q}{r_0} - \frac{Q}{r_1} + \frac{Q'}{r_2}.$$

The potential of the shell would be

$$V_1 = \frac{Q - Q + Q'}{r_2}.$$

The difference of potential between them would be

$$V_0 - V_1 = 9 = Q\left(\frac{1}{r_0} - \frac{1}{r_1}\right) = Q\left(\frac{1}{20 \times 21}\right)$$

$\therefore$ $Q = 3780$ and $C = 420$ as before. (See section 76.)

226. *Arrangement of battery cells for maximum current.*

The solution of this problem applies to either primary, or storage cells, or to a number of dynamos.

Suppose we have n cells arranged in n' lines of n'' cells each. Let the electromotive force of each cell be e and let the cell resistance be r. Let the interpolar or external resistance be R'. We will consider R' to be the resistance of the load.

The battery *E. M. F.* is $n' e$, and its resistance is $\frac{n' r}{n''}$. By Ohm's law

$$i = \frac{n' e}{\frac{n'}{n''} r + R'} = \frac{e\, n'\, n''}{r\, n' + R'\, n''} \quad \ldots\ldots\ldots (407.)$$

In this equation we have two independent variables, n' and n'', and i is a function of these variables. Equation (407) may be represented by a surface Fig. 141. If

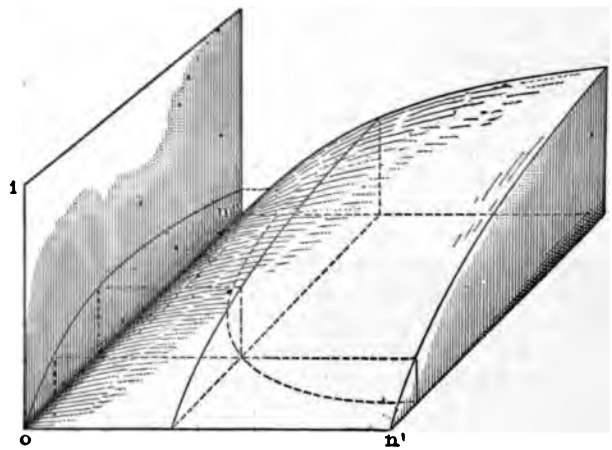


Fig. 141.
Condition of maximum current from a battery.

we pass a plane through this surface at $R \perp$ to the axis n', the equation of the intersection will be obtained from Eq. (407) by making n' constant. We shall then have

$$i = \frac{a\,n''}{b + R'\,n''} \quad \dots\dots\dots\dots\dots\dots (408)$$

where $a = e\,n'$ and $b = r\,n'$

This is the equation of an equilateral hyperbola. If $n'' = \infty$, $i = \frac{a}{R'} = \frac{en'}{R'}$. This determines one asymptote of the curve. The other is evidently determined by making the denominator of (408) equal to zero or $n'' = -\frac{b}{R'} = -\frac{rn'}{R'}$. The distances to these asymptotes become zero at the origin, where $n' = 0$. This determines two planes which contain all asymptotes, and the intersection of which may readily be found. Similarly if n'' be made constant, Eq. (407) becomes

$$i = \frac{h\,n'}{r\,n' + k} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (409)$$

where $h = e\,n''$ and $k = R'\,n''$. This equation is of precisely the same form as (408). It is the equation of any plane section at right angles to the axis n''. The asymptotes of this hyperbola can be located in the same manner. Neither of these sections contains a maximum for finite values of n'' and n'. If we now fix n then we shall have

$$n'\,n'' = n \qquad\qquad (410).$$

This is the equation of an equilateral hyperbola having the axes n' and n'' as asymptotes. The current from this battery of n cells, will be represented by the ordinates of points on the surface represented by Eq. (407), which lie directly over the hyperbola represented by Eq. (410). When either n' or n'' becomes very large the ordinates of such points approach zero. (See Fig. 141.) If we combine Eq. (410) with Eq. (407) by the elimination of n', we shall have the equation of the projection of the same points on the surface, on the plane of i, n''. The resulting equation is

$$i = \frac{e\,n\,n''}{rn + R'\,n''^{\,2}} \qquad\qquad (411).$$

The projection itself is the curve shown on the vertical co-ordinate plane in Fig. 141.

Mathematically this value of i approaches zero when n'' is large, and it is also zero when $n'' = 0$. It has a maximum when

$$\frac{di}{dn''} = \frac{(rn + R'\,n''^2)\,en - 2\,R'\,en\,n''^2}{(R'n + R'n'')^2} = 0.$$

$$\text{or}\ \frac{rn'}{n''} = R' \qquad\qquad (412).$$

The battery resistance is then equal to the external resistance. The same result will be reached by combining (410) with (407) by the elimination of n''. The resulting equation will be

$$i = \frac{e\,n\,n'}{r\,n'^{\,2} + R'n} \qquad\qquad (413).$$

This equation is the same in form as (411). It is the equation of the projection of points lying over the hyperbola (410) upon the other co-ordinate plane. Equations (410) (411) and (413) are projections of the points of

ting to the secondary circuit becomes too small to show distinctly in the figure when K is as large as 20.

The last equation shows that the current in the secondary circuit is the same as would be produced by a fixed potential $\frac{V_1}{K}$, working through a resistance equal to that of the secondary circuit, increased by $\frac{r_1}{K^2}$ which is equal to that of the secondary coil.

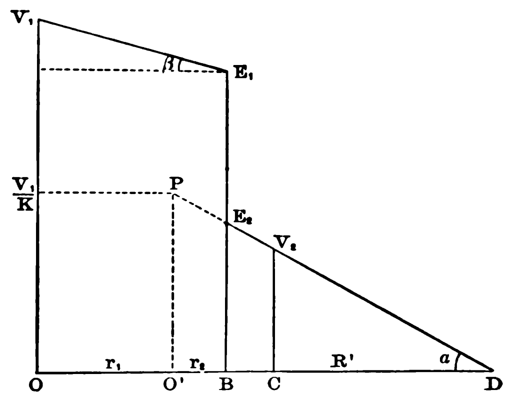


Fig. 144.
Potential diagram for transformer circuit.

In Fig. 144 $\overline{O\ V_1}$ represents the supply potential V_1. $\overline{O\ B}$ is the resistance of the primary coil and $B\ C$ that of the secondary. Since in this figure $K = 2$, $B\ C = \frac{1}{4}\ \overline{O\ B}$. $\overline{B\ E_1}$, and $\overline{B\ E_2}$ are the *E. M. F.'s* of the primary and the secondary coils, where in the figure $E_1 = 2\ E_2$. $B\ D$ represents the resistance of the secondary circuit, and $B\ O'$ is the virtual increase, $\frac{r_1}{K^2}$, in the denominator of the last equation. The point V_1 is a fixed point in the diagram for all changes in load R'. This condition is determined by the regulation at the power-house. The point P is also a fixed point, as the last equation shows. If the load increases, and $\overline{C\ D} = R'$, becomes less, the slope of P D = *tan a* = i_2 becomes greater. The terminal potential V_2 and the secondary

The economy of operation is

$$\eta = \frac{w'}{w} = \frac{n'' R'}{n' r + n'' R'} \text{ ....(416)}$$

Or

$$n'' = \frac{r}{R'} \frac{\eta}{1 - \eta} n' \text{ ....(417).}$$

If $r = 0.008$ ohm and $R' = 0.2$ ohm

$$n'' = 0.04 \frac{\eta}{1 - \eta} n'.$$

This equation represents a series of right lines on the plane of n', n'' of Fig 141; all of which pass through the origin Fig. 142. If $\eta = 0.9$; $n'' = 0.36\, n'$. If $\eta = 0.8$; $n'' = 0.16\, n'$. The lines for various values of η may thus be plotted on the plane n', n'', and the hyperbolæ corresponding to various values of n may be plotted across them. The intersections of the two sets of lines will determine the efficiency of operation of a battery of any size arranged in any manner, if r and R' have been fixed. If e be now fixed, the surface of Fig. 141 is fixed, and the ordinate corresponding to any point determined as above will give the current.

Fig. 142 gives the lines representing economy of operation from 0.98 to 0.5, the axis n' representing $\eta = 0$ and the axis n'' representing $\eta = 1.00$. The same figure gives curves of total power developed, computed from (414). In these curves, the constants assumed are $r = 0.008$ ohm, $R' = 0.2$ ohms, $e = 2.0$ volts.

Eq. (414) then becomes

$$n'' = \frac{0.008\, w\, n'}{4\, n'^2 - 0.2\, w}.$$

If we make $w' = 160{,}000$ or 160 *K. W.* then if $n' = 100$ cells, the number of lines of such cells required will be

$$n'' = \frac{0.008 \times 160{,}000 \times 100}{4 \times 10000 - 0.2 \times 160{,}000} = 16.$$

By reference to Fig. 142 it will be seen that the economy of operation is $\eta = 0.8$. The number of cells required is 1600. If a single line of 100 cells arranged in series is to be used for this work, they must be 16

circuit is somewhat shorter and the number of windings in both coils is correspondingly diminished. The net section of the core multiplied by the number of windings is the same, in both cases. By reason of the errors of graphical solution, this last check may not be realized with minute precision, nor is this necessary, since all of the dimensions must be adjusted to fit commercial sizes.

The secondary coil must be wound in 8.1 layers of 6.4 windings each. Therefore the dimensions of the openings in the core-plates are $h = 2.81$, and $h' = 2.81 + 6.4 \times 0.3429 = 5.0$. The volume of the iron becomes by Eq. (434)

$$v = 2\,(4\ 99 \times 7.18 - 2.18 \times 5.0)\,\frac{9 \times 2.18}{1.2} = 815$$

cubic centimetres.

The hysteresis loss per cubic centimetre is the same as in the other case. The total hysteresis loss in the core will, therefore, be greater than in the former case by reason of an increase in the volume of the iron from 750 to 815 cubic centimetres. This loss will be $\frac{815}{750}$ $58 = 1.086 \times 58 = 63$ watts.

The change in the eddy-current loss is not so easily determined. It is evident that a greater proportion of the copper is inside the iron than in the former case, and the copper may, therefore, have a higher temperature. This would increase the copper loss. This would also heat the iron to a higher temperature, which would diminish eddy current loss, since it would diminish the conductance c of the iron. At the same time, the radiating surface of the iron is increased. This indicates the nature of the changes due to a change in the section of the iron, a full discussion of which would be more appropriate in an engineering treatise.

We shall assume that the resulting temperature of the iron is the same as in the former case.

The factor $\frac{m^2}{m^2+1}$ of Eq. (401) is 1.00 as before. The eddy-curre[illegible] [illegible] would then be increased by reason of the inc[illegible] [illegible]me of the iron, and would be $\frac{815}{750}$

equal economy. They may also be considered as contour lines of a surface representing η. The useful power w' developed in the working resistance is represented by Eq. (415). It may also be represented by a surface. The value of $w' = \eta\, w$ may, however, be found from Fig. 142 by finding the product $w\,\eta$ at any point.

The equation of any curve representing useful power as deduced from (415) is

$$n'' = \frac{r\,n'}{e\,n'\sqrt{\dfrac{R'}{w'}} - R'} \quad \text{.........................(418).}$$

Substituting the special values used in Fig. 142

$$n'' = \frac{0.008\,n'}{2\,n'\sqrt{\dfrac{0.2}{w'}} - 0.2}.$$

The equation for the curve representing a useful power of 40,000 watts or 40 kilowatts is

$$n'' = \frac{0.008\,n'}{0.004472\,n' - 0.2}.$$

Values of n' less than 44 give negative values of n''. This locates an asymptote of the curve. This asymptote is determined by making the denominator of (418) equal to zero. We thus have $n' = \dfrac{\sqrt{R'\,w'}}{e}$ which for $w' = 40{,}000$ gives $n' = 44$. Giving n' greater values we compute n'' and may thus determine contour lines for the surface w' like those given for w in Fig. 142.

It will be observed that the power curves on the surfaces w and w' have one of their asymptotes in common. When n'' becomes very large, the battery resistance approaching zero, the economy approaches unity. We have then from (414) and (415)

$$w_{n''=\infty} = \frac{e^2\,n'^2}{R'} \text{ or } n' = \frac{\sqrt{R'\,w}}{e}$$

$$w'_{n''=\infty} = \frac{e^2\,n'^2}{R'} \text{ or } n' = \frac{\sqrt{R'\,w'}}{e}.$$

This locates one of the asymp

power curves of Fig. 142. Similarly if n' becomes very large, approaching an infinite value,

$$w_{n'=\infty} \frac{e^2 n' n''}{r} \text{ or } n'' = \frac{r w}{e^2 n'}.$$

When $n' = \infty$, $n'' = 0$, or the axis n' is an asymptote of all power curves of Fig. 142. Similarly

$$w'_{n'=\infty} = \frac{e^2 R' n''^2}{r^2} \text{ or } n'' = \frac{r}{e}\sqrt{\frac{w'}{R'}}.$$

These other asymptotes for the power curves on the surface representing useful power depend, therefore, on w' and R'. The axis n' is an asymptote only when w' is zero.

228. *To determine the mechanical equivalent of heat.* The apparatus consists of three glass tubes intended for Liebig's condensers. They are coupled together as

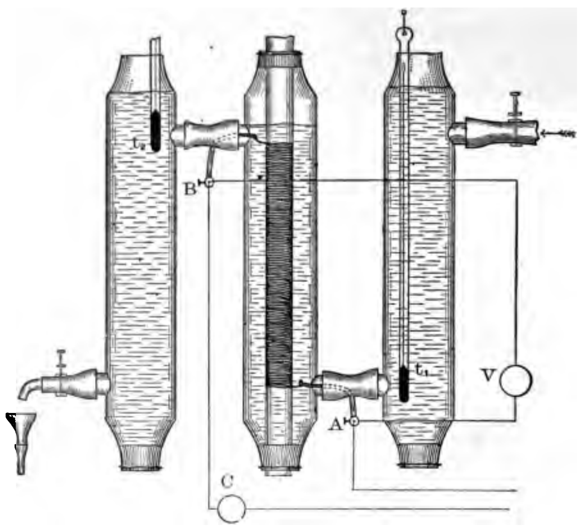


Fig. 143.
Determination of the number of heat units in a joule.

shown in Fig. 143. The condensing tube of the central condenser which is filled with cotton is wound with a single layer of fine wire having cotton insulation. There are about 500 windings, which connect with large leading-

in wires led through the rubber tubing to binding screws A B. Thermometers placed at the entrance and discharge of the central condenser, measure the temperature of water which flows through the three vessels at any constant velocity. The velocity of inflow and of outflow can be regulated by screw clamps on the rubber pipes which serve for supply and discharge. The supply water is from an overflowing tank fed from the hydrant, and having masses of ice within, in order to reduce the temperature of the inflowing water. A current is passed through the fine wire. This current, and the potential difference between A and B, are measured in amperes and volts with the amperemeter C and the voltmeter V. The power applied to the wire helix is $v\ i$ watts. After the temperatures have become constant, the thermometers are read. The reading of the thermometers at the entrance should be as much below that of a thermometer in air, as that of the thermometer at the discharge is above it. Weigh the amount of water flowing through in any time and compute the number of grammes m flowing through per second. The heat received by the water per second is then $m\ (t_2 - t_1)$. Then

$$J\ (t_2 - t_1)\ m = i\ v$$

$$J = \frac{i\ v}{(t_2 - t_1)m}.$$

In an actual determination the temperatures were, entrance, 12.°41 C.; discharge, 16.°22. The rise in temperature was, therefore, 3.°81 C. The amount of water flowing per second was 15.75 grammes. The heat received by the water was at the rate of $15.75 \times 3.81 = 60.01$ calories per second. The current was 6.24 amperes. The difference of potential on the terminals was 39.7 volts. The power applied was, therefore, $39.7 \times 6.24 = 247.728$ watts. The mechanical equivalent of heat is, therefore, $J = \frac{247.728}{60.01} = 4.01$ practical units. In this determination the temperature of the room was 20°.4 C.; so that some of the heat imparted to the water came from the room. This would make the heat too great, and would make J somewhat too small. See *section (132.)*

If d = the diameter of the covered wire, assuming it to be wound in two layers

$$l = (2b \times 2h + 8d)\, N \ldots\ldots\ldots\ldots (439).$$

Eliminating A in (437) and (438)

$$bhnN = 3.9 \times 10^5 \ldots\ldots\ldots\ldots (440).$$

Dividing (440) by (439)

$$\frac{bh}{b + h + 4d} = \frac{7.8 \times 10^5}{n\,l} \ldots\ldots\ldots (441).$$

We may now assume a relation between b and h which will determine the shape of the rectangular section of the iron core, and put

$$b = q\,h \ldots\ldots\ldots\ldots (442).$$

Eq. (441) then becomes

$$\frac{h^2\,q}{h\,(q + 1) + 4\,d} = \frac{7.8 \times 10^5}{n\,l} = M$$

or

$$h^2 - M\,\frac{q + 1}{q}\,h = \frac{4\,M\,d}{q}.$$

Solving for h,

$$h = \frac{1}{2}\,M\,\frac{q + 1}{q} \pm \sqrt{\frac{4\,M\,d}{q} + \frac{1}{4}\,M^2\left(\frac{q + 1}{q}\right)^2} \quad (443).$$

If we make the number of revolutions per second $n = 15$, or 900 per minute, and make $r_s = 50\ R_1$ (y being therefore 50), then by the table in section 231, $r_a = 0.048$ ohms; $r_s = 76.5$ ohms; $i_a = 65.3$ amperes; and $E = 101.1$ volts. If the armature wire is to carry 4 amperes to the square millimetre, then by the table of section 232, the section and length of the armature wire will be $s = 0.0816$ square centimetre, and $l = 7355$ centimetres. The value of M then becomes

$$M = \frac{7.8 \times 10^5}{15 \times 7355} = 7.070.$$

The diameter of the wire of the armature will be 0.322 square centimetre (since $s = 0.0816$) and the diameter of the covered wire will be about equal to 0.37 $= d$. Eq. 443 then becomes

$$h = 3.54\,\frac{q + 1}{q} \pm \sqrt{\frac{10.46}{q} + 12.5\left(\frac{q + 1}{q}\right)^2} \ldots\ldots (444).$$

If q be taken less than unity, the armature will be of the disc form, like that of Schuckert. If $q = 1$ the iron core will have a square section, while if $q > 1$ the radial thickness h of the core will be less than the breadth measured parallel to the shaft. If the iron core is of wire the square section is open to the objection that the lines of the field do not reach the iron windings nearest to the shaft, as readily as those on the outside of the ring. When iron discs are used this consideration is of less importance. The following table gives the values of h, b and N from (444), (442) and (440), for various values of q.

q	h	b	N
1	14.87	14.87	118
2	11.08	22.16	106
3	9.78	29.34	91
4	9.12	36.48	78
∞	7.08	∞	0

For an armature as large as the one corresponding to $q = 4$, the shaft should have a diameter of about 7 cm. The sleeve around the shaft and the space required for the return wires will require not less than 2.5 centimetres of radial distance. The inner diameter of the iron ring cannot be less than 12 centimetres. The outer diameter would therefore be 30 centimetres. It is then evident that the copper wires can be accommodated in one layer even on the inside of the ring. The preliminary assumption made was that the wires should be in two layers.

This, however, does not materially affect the result. It would, however, be advisable to make a final computation on the same lines as the one here given, correcting the assumptions in accordance with the information which the preliminary computation reveals, and adjusting the *dimensions* to commercial sizes.

The heat generated in this armature will be

1. Copper loss,

$$w' = r_a i_a^2 = 0.048 \times (65.3)^2 = 205 \text{ watts.}$$

2. Hysteresis loss per cubic centimetre by Eq. (396),

$$\frac{w''}{v} = \frac{0.002 \times 15 \times 16000}{10^7} = 0.016 \text{ watts.}$$

The net volume of the iron core is

$$v = 0.8 \pi \left(\frac{900 - 144}{4}\right) 36 = 17000 \text{ cc.}$$

The total hysteresis loss is, therefore,

$$w'' = 0.016 \times 17000 = 272 \text{ watts.}$$

3. Eddy-current loss.

Assuming a running temperature of 50° C. the conductance of soft iron would be $c = 8.65 \times 10^4$ (see section 133). Hence by Eq. (400) assuming the diameter of the iron wire to be 0.14 cm.

$$\frac{w'''}{v} = \frac{\pi^2 \times 16000^2 \times 15^2 \times 8.65 \times 10^4 \times 0.14^2}{16 \times 10^{16}} = 0.0060.$$

The eddy-current loss will therefore be

$$w''' = 0.006 \times 17000 = 102 \text{ watts.}$$

The total armature loss is therefore

$$w_a = 205 + 272 + 102 = 579 \text{ watts.}$$

The surface of the armature not including the inner surface of the ring is 4580 square centimetres. This is 7.9 square centimetres to the watt of loss. Experience shows that 7.5 centimetres to the watt of loss is sufficient to prevent overheating in armatures run at ordinary speeds.

The loss in the field magnet coil is $i_s^2 r_s = (1.28)^2 \times 756. = 128$ watts.

The total losses in the machine will be therefore 579 + 128 = 707 watts. The useful power delivered to the load R is $i^2 R = 64^2 \times 1.53 = 6267$ watts. The total power developed is therefore 707 + 6267 = 6974 watts. The efficiency of the machine is therefore $\frac{6267}{6974} =$ *89.8 %.* In this determination brush, journal and air *friction* are not included.

INDEX.

THE NUMBERS REFER TO THE PAGES.

A.

B.

C.

D.

F.

G.

H.

I.

J.

L.

M.

Q.

R.

S.

T.